Status of NLO Antenna Showers

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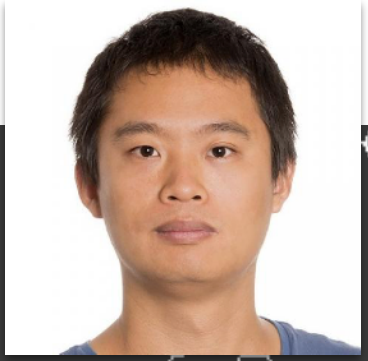
Based on:

- 2013: $(q\bar{q} \to)\, qg\bar{q}$ at NLO [HLS, JHEP 1310 (2013) 127]
- 2016: Framework for full FSR at LC [LS, PLB771 (2017) 59]

… + work in progress …



June 2017

CERN TH Institute on Physics at the LHC

MOTIVATION

Motivation: not *a priori* to do N(N)LL evolution, nor do we at this point claim that we do.

Wanted to see if we could use experience with tree- and one-loop matrix-element corrections in showers → derive & implement a set of self-consistent 2nd-order corrections to our shower kernels

To be used all throughout the shower.

Expect this → N(N)LL evolution for some set of observables but not the focus of our work so far.

Many (interesting) questions remaining

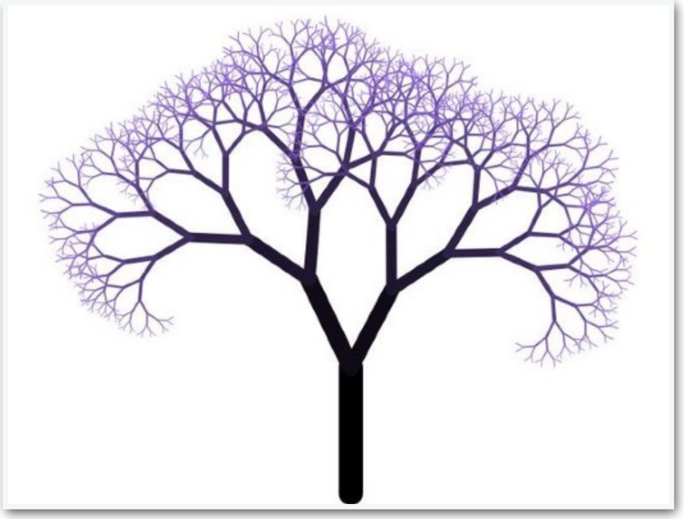
Initial-State Radiation (interface with PDFs), radiation in resonance decays, heavy quarks, QED radiation, merging with fixed order

Matrix-Element Corrections

Regard shower as generating approximate weighting of (all) n-parton phase space(s) ~ sums over radiation functions times Sudakov factors

Captures universal leading singular structures, but not subleading or process-dependent terms → impose M.E. corrections order by order.

- Used extensively in PYTHIA to correct first emission in all SM decays, most BSM ones, and for colourless boson production + ISR
- Is the basis for the real corrections in POWHEG
- Generalised to multiple emissions in VINCIA



Tree-level, 1 emission:
Sjöstrand & Miu PLB449 (1999) 313-320
Sjöstrand & Norrbin Nucl.Phys. B603 (2001) 297

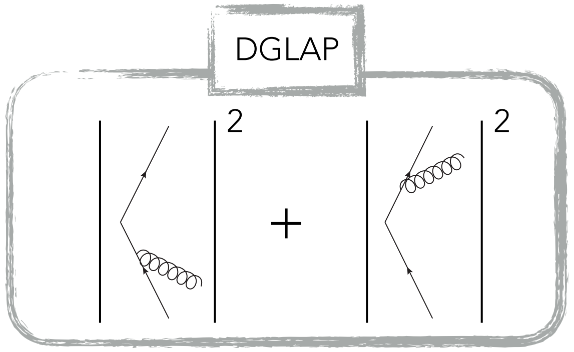
One-loop, 0 emissions: Nason (POWHEG), JHEP 0411 (2004) 040

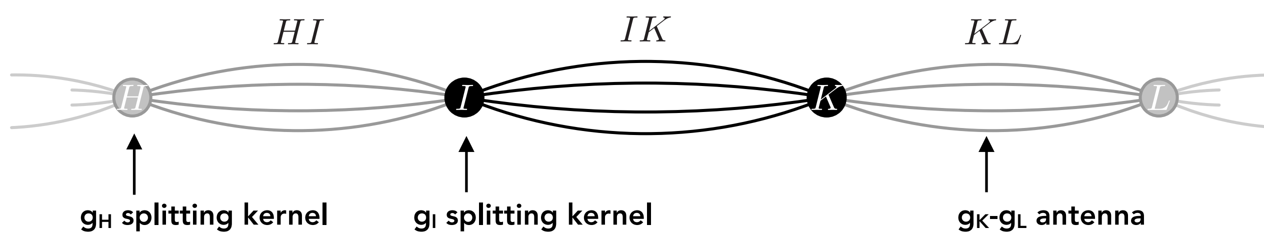
Tree-level, 1:N emissions: Giele, Kosower, PZS, PRD84 (2011) 054003
One-loop, 0:1 emission: Hartgring, Laenen, PZS, JHEP 1310 (2013) 127

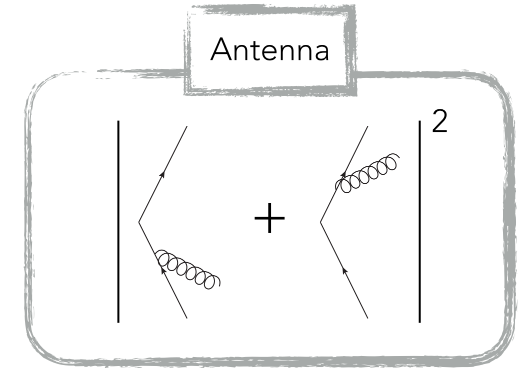
Shower contains correct singularities for all single-unresolved limits ⟷ Corrections nonsingular @ NLO

True for any (coherent) shower model.

(COHERENCE : DGLAP VS ANTENNAE)







DGLAP: based on collinear limits

- Each parton treated as an ~ independently radiating monopole, $P(z)/Q^2$
- Misses soft-limit coherence, already at leading (dipole) level
  - Ang. ord. (or vetos) → correct soft limit when summed over azimuth
    - (But phase-space *distributions* of emitted gluons still not point-by-point correct)
- Matrix-Element corrections can restore exact coherence point-by-point, up to order applied
  - E.g., a DGLAP shower could be improved by "dipole corrections" at all orders (but we have dipole showers)

Antenna evolution: each LC-connected **parton pair** ~ radiating dipole-antenna

- Splittings fundamentally 2→3 instead of 1*→2 Gustafson & Petterson: NPB306 (1988) 746-758

Antenna Factorisation of:

- **Phase Space:** Lorentz-invariant on-shell 2→3 phase-space maps exact over all of phase space, not just limits.
- **Amplitudes:** Correct in collinear **and** soft limits (to all orders): Kosower PRD71 (2005) 045016
  - Collinear limits → $P(z)/Q^2$
  - Soft limits → eikonal factors $\frac{2s_{13}}{s_{12}s_{23}}$
    - Point-by-point coherence (at LC; higher colour multipoles suppressed by $1/N_C^2$)

Complete set of NNLO final-state antenna functions: Gehrmann-de Ridder, Gehrmann, Glover JHEP 0509 (2005) 056

Antenna phase-space maps obey exact nesting

$$\mathrm{d}\Phi_{n+1} = \mathrm{d}\Phi_n \times \mathrm{d}\Phi_{\mathrm{ant}}$$ (one clustering)

↳ Generalisation to many possible clusterings: $\mathrm{d}\Phi_{n+1} = \sum_{i=1}^{n_{\mathrm{ant}}} f_i \, \mathrm{d}\Phi_{\mathrm{ant,i}} \mathrm{d}\Phi_n^i$

Ordering/partitioning function (global or sector)

Sector showers: f_i = partition of unity (x strong-ordering)

~ WINNER-TAKES-ALL JET ALGORITHMS

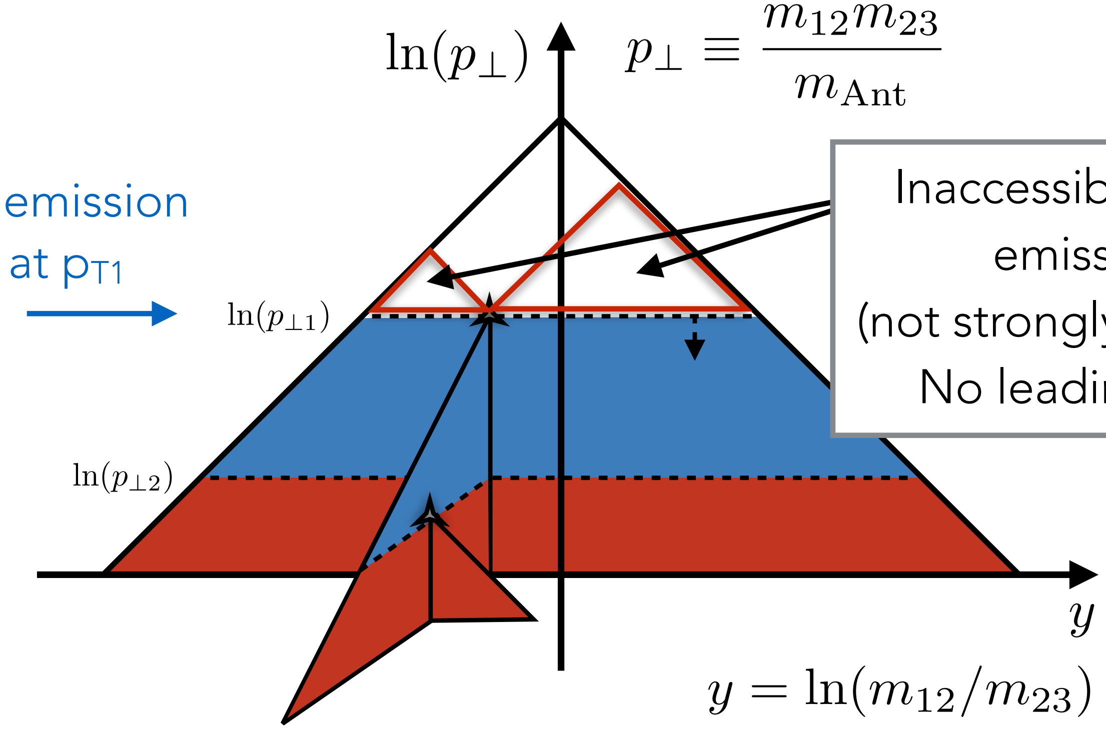
Lopez-Villarejo & PZS: JHEP 1111 (2011) 150

Global showers: f_i = multiple cover (x strong-ordering)

ANTENNA FUNCTIONS ***SUM*** TO TOTAL SINGULARITIES

→ Can cover all of phase space; but do we?

For a general shower ordering variable, the 2→4 (and higher) phase spaces exhibit regions with all f_i = 0 (no ordered paths) → inaccessible

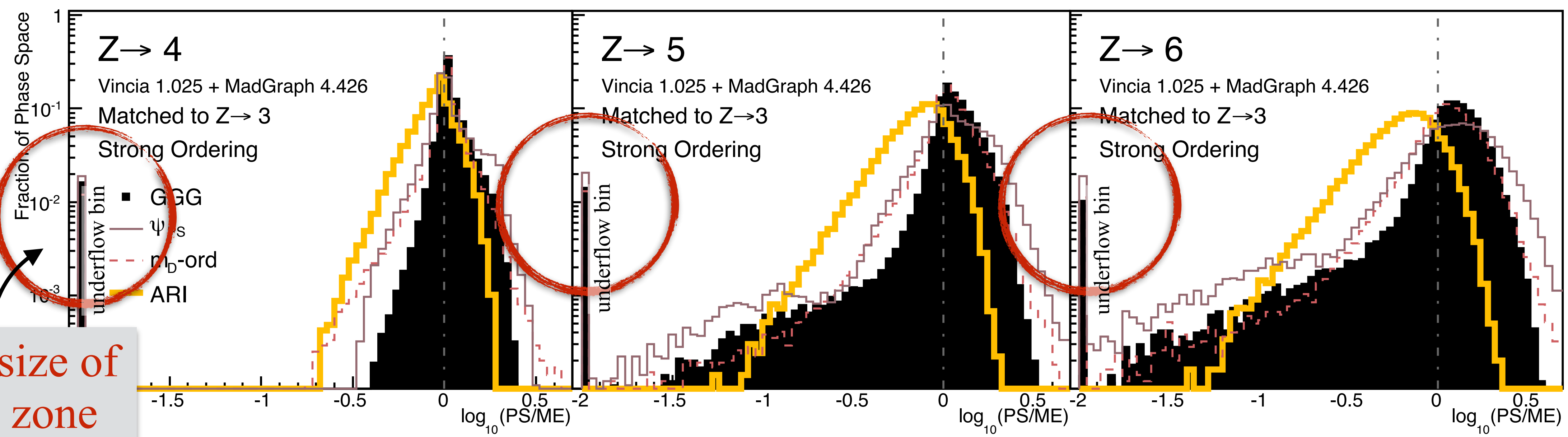


(a) Strong Ordering

HOW BIG ARE THESE REGIONS?

Giele, Kosower, PZS: PRD84 (2011) 054003

Flat scans of N-parton phase space (RAMBO)



Total size of dead zone ~ 2% of PS

$$R_N = \log_{10}\left(\frac{\mathrm{Sum(shower\text{-}paths)}}{|M_N^{(\mathrm{LO,LC})}|^2}\right)$$

PS = shower expanded to tree level, summed over all ordered paths

ME = LO matrix element (MADGRAPH @ leading colour)

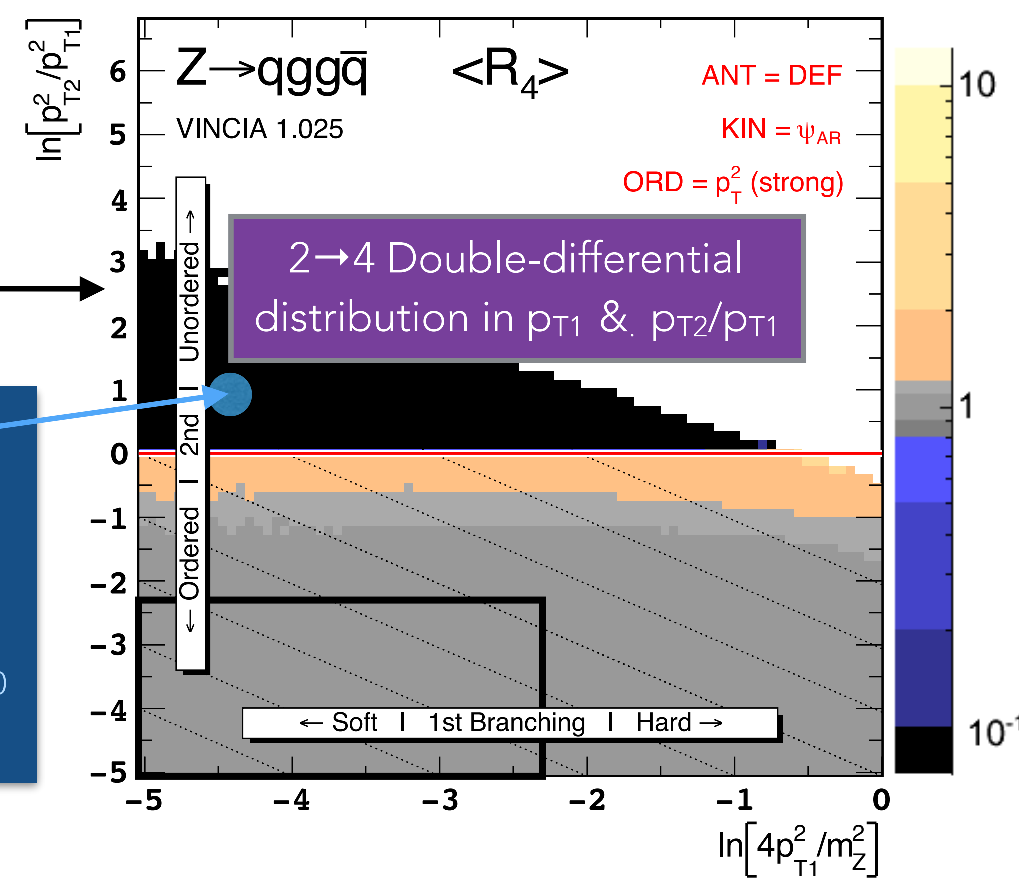
$Q_0 = 91\,\mathrm{GeV}$

$p_{T1} = 5\,\mathrm{GeV}$

$p_{T2} = 8\,\mathrm{GeV}$

Unordered with $p_{T2} << Q_0$

"Double Unresolved"



2→4 Double-differential distribution in p_{T1} & p_{T2}/p_{T1}

THE SOLUTION THAT WORKED AT LO

Wanted starting point for (LO) matrix-element corrections over **all** of phase space (good approx → small corrections)

Allow newly created antennae to evolve over their full phase spaces, with suppressed (beyond-LL) probability: **smooth ordering**

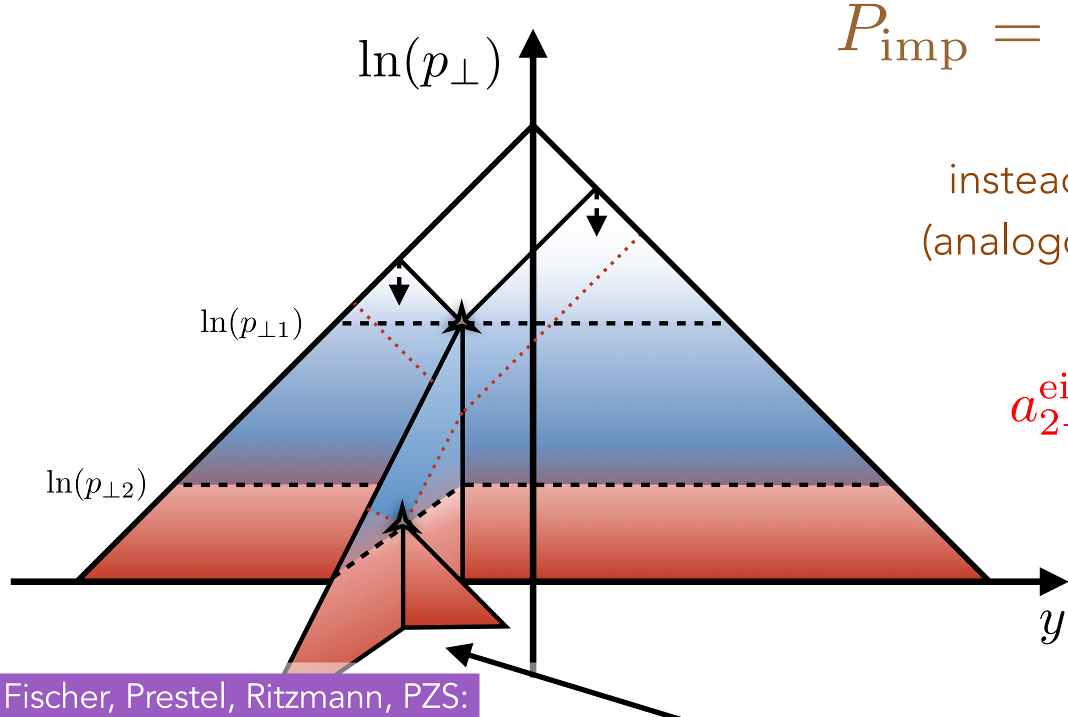
Giele, Kosower, PZS: PRD84 (2011) 054003

$$P_{\mathrm{imp}} = \frac{p_{\perp n-1}^2}{p_{\perp n-1}^2 + p_{\perp n}^2}$$

$\to 1$ for $p_{\perp n} \ll p_{\perp, n-1}$
$\to 1/2$ for $p_{\perp n} \sim p_{\perp, n-1}$
$\to 0$ for $p_{\perp n} \gg p_{\perp, n-1}$

instead of strong ordering
(analogous to POWHEG hfact)

$$a_{2\to 4}^{\mathrm{eik}} \sim \frac{1}{p_{\perp n-1}^2} P_{\mathrm{imp}} \frac{1}{p_{\perp n}^2} \propto \begin{cases} 1/p_{\perp n}^2 & \text{ordered} \\ 1/p_{\perp n}^4 & \text{unordered} \end{cases}$$



Figures from Fischer, Prestel, Ritzmann, PZS: EPJC76 (2016) 11, 589

(b) Smooth Ordering

Leading Logs unchanged

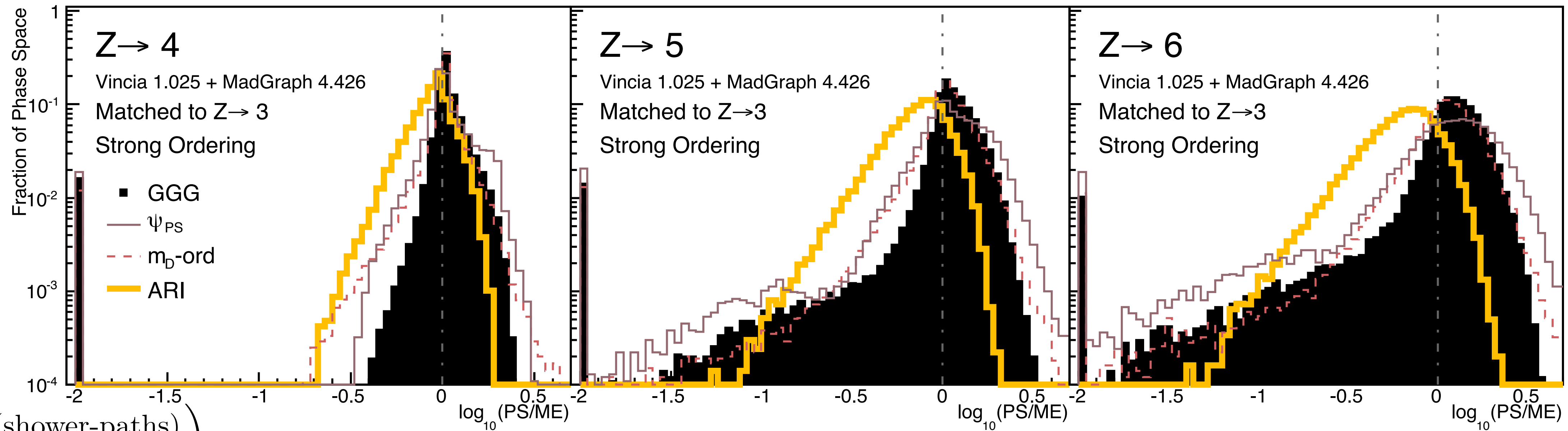
Fischer, Prestel, Ritzmann, PZS: EPJC76 (2016) 11, 589

$$-\ln\Delta \propto \int_{p_\perp^2}^{m^2} \frac{1}{1+\frac{q_\perp^2}{Q_\perp^2}} \frac{dq_\perp^2}{q_\perp^2} \ln\left[\frac{m^2}{q_\perp^2}\right] \sim \left(\frac{1}{2}\ln^2\left[\frac{Q_\perp^2}{p_\perp^2}\right] + \ln\left[\frac{Q_\perp^2}{p_\perp^2}\right]\ln\left[\frac{m^2}{Q_\perp^2}\right]\right)$$

Note: this conclusion appears to differ from that of Bellm et al., Eur.Phys.J. C76 (2016) no.1
My interpretation is that, in the context of a partonic angular ordering, they neglect the additional rapidity range from the extra origami folds

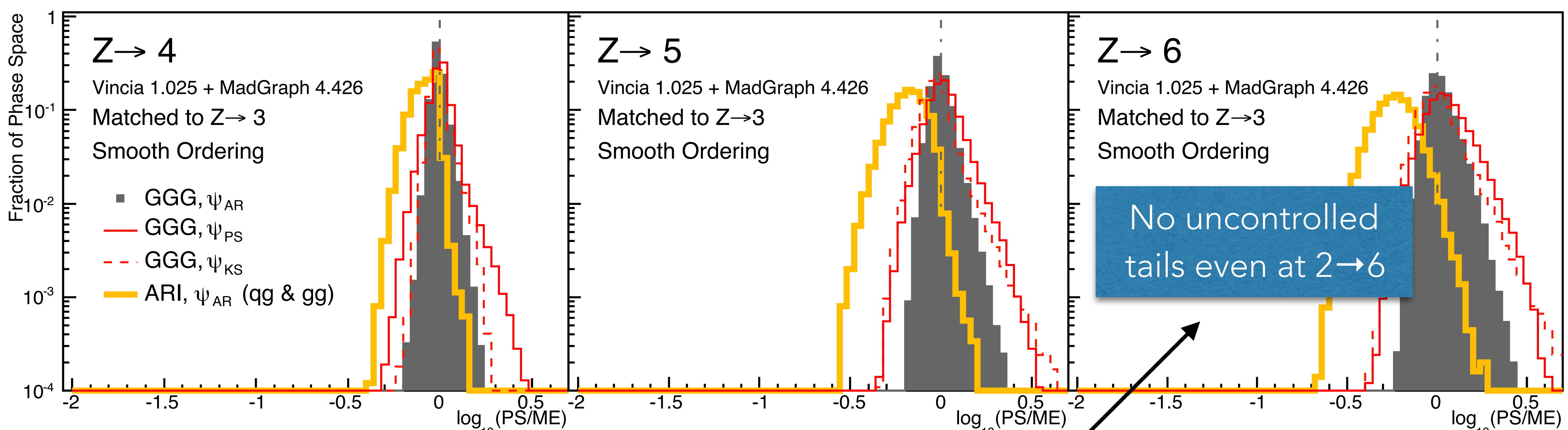
SMOOTH ORDERING: AN EXCELLENT APPROXIMATION (at tree level)

Strong



$$R_N = \log_{10}\left(\frac{\text{Sum(shower-paths)}}{|M_N^{(\text{LO,LC})}|^2}\right)$$

Smooth



No uncontrolled tails even at 2→6

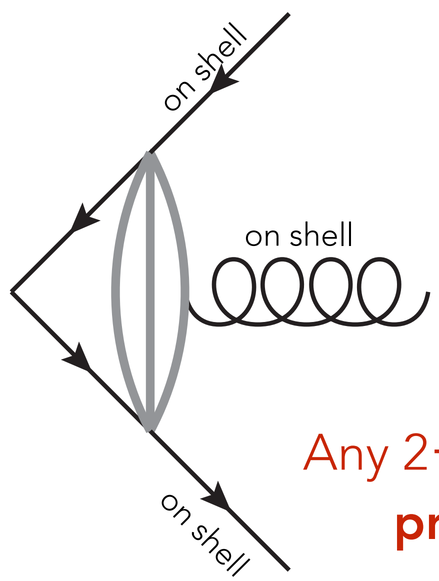
Even after three sequential shower emissions, the smooth shower approximation is still a very close approximation to the matrix element **over all of phase space**

(WHY IT WORKS?)

The antenna factorisations are **on shell**

n on-shell partons → **n+1** on-shell partons

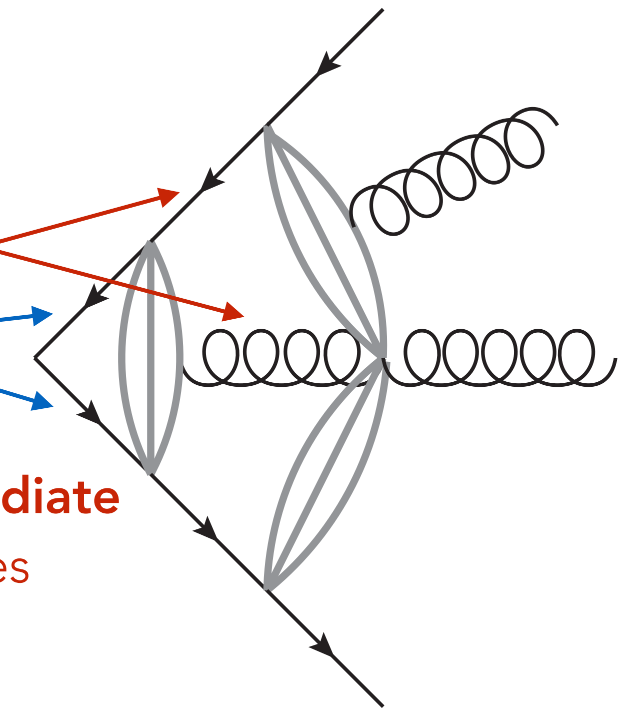
In the first 2→3 branching, final-leg virtualities assumed ~ 0



Strong Ordering:

these virtualities

small compared to these virtualities



Any 2→4 Feynman diagrams we draw will involve **intermediate propagators** with virtualities of order the last p_T^2 scales

Cannot be neglected in unordered part of phase space

Interpretation: off-shell effect

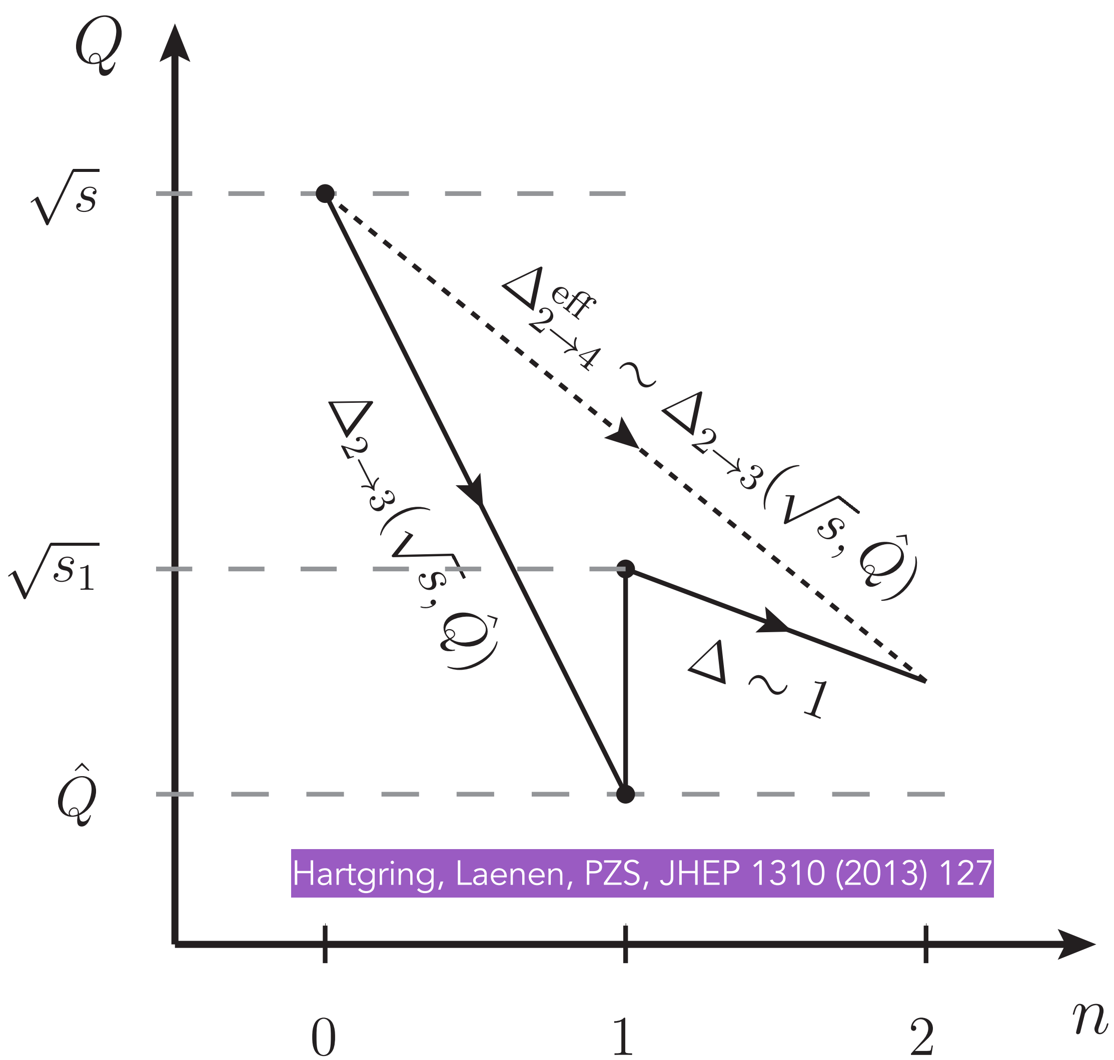
$$\frac{1}{2p_i \cdot p_j} \to \frac{P_{\mathrm{imp}}(n \to n+1)}{2p_i \cdot p_j} = \frac{1}{2p_i \cdot p_j + \mathcal{O}(p_{\perp n+1}^2)}$$

Good agreement with ME → good starting point for 2→4

SOMETHING ROTTEN?

Smooth ordering: nice tree-level expansions (small ME corrections) ⇨ good 2→4 starting point

But we worried the Sudakov factors were "wrong" ⇨ not good starting point for 2→3 virtual corrections? Not good exponentiation?



For unordered branchings (e.g., double-unresolved) effective 2→4 Sudakov factor effectively → LL Sudakov for intermediate (unphysical) 3-parton point

DIRECT 2→4

Li & PZS: PLB771 (2017) 59

Redefine the shower resolution scale

For unordered 2→4 paths: scale of **2nd** branching defines resolution

The intermediate on-shell 3-parton state is merely a convenient stepping stone in phase space but is in reality highly off shell ⇨ integrate out

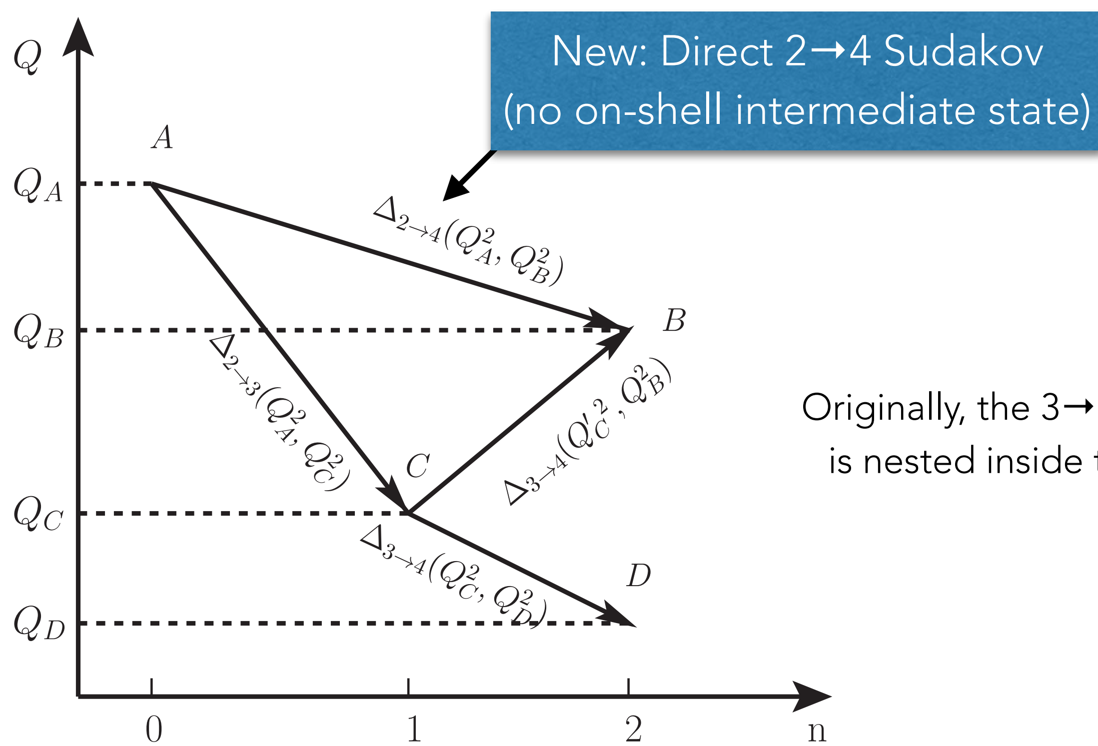


Figure 1: Illustration of scales and Sudakov factors in strongly ordered (ACD), smoothly (un)ordered (ACB), and direct 2 → 4 (AB) branching processes, as a function of the number of emitted partons, n.

Interchange order of integrations
$Q_{2\to3} \leftrightarrow Q_{3\to4}$

Unordered phase space: $Q_4 > Q_3$

$$\int_0^{Q_0^2} dQ_3^2 \int_{Q^2}^{Q_0^2} dQ_4^2\, \Theta(Q_4^2 - Q_3^2)\, f(Q_3^2, Q_4^2) = \int_{Q^2}^{Q_0^2} dQ_4^2 \int_0^{Q_4^2} dQ_3^2\, f(Q_3^2, Q_4^2)\,,$$

Originally, the 3→4 phase space is nested inside the 2→3 one

Now the intermediate (unordered) scale is integrated over for each value of Q_4

for a generic integrand, f, with the result:

$$\Delta_{2\to4}(Q_0^2, Q^2) = \exp\left[-\sum_{s\in a,b} \int_{Q^2}^{Q_0^2} dQ_4^2 \int_0^{Q_4^2} dQ_3^2 \int_{\zeta_{4-}}^{\zeta_{4+}} d\zeta_4 \int_{\zeta_{3-}}^{\zeta_{3+}} d\zeta_3 \frac{|J_3 J_4|}{(16\pi^2)^2 m^2 m_s^2} \int_0^{2\pi} \frac{d\phi_4}{2\pi} R_{2\to4}\, s_3\, s_3'\right], \qquad (11)$$

Product of 2→3 functions

Jacobian for dLIPS → $dQ_3 dQ_4 d\zeta_3 d\zeta_4$

2→4 MEC

Split the 2→4 phase space into non-overlapping sectors

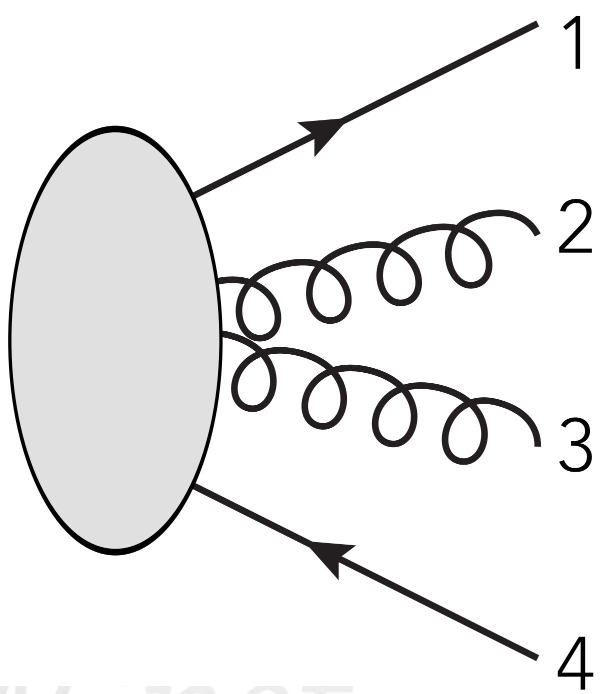
Fully unordered (inacessible to iterated 2→3)

⇨ add new "direct" 2→4 branchings without risk of double-counting

Rest of phase space (accessible to at least one ordered 2→3 path)

Unitarity (Sudakov exponentials and virtual corrections): want to sum inclusively over the "least resolved" degree of freedom

Classify according to what a jet algorithm (with shower evolution parameter as clustering measure) would do. E.g., for a (colour-connected) double-emission:



$$p_{\perp 2} \equiv \frac{m_{12} m_{23}}{m_{123}}$$

$$p_{\perp 3} \equiv \frac{m_{23} m_{34}}{m_{234}}$$

A jet clustering algorithm (ARCLUS) would grab the smallest of these p_T values, and cluster

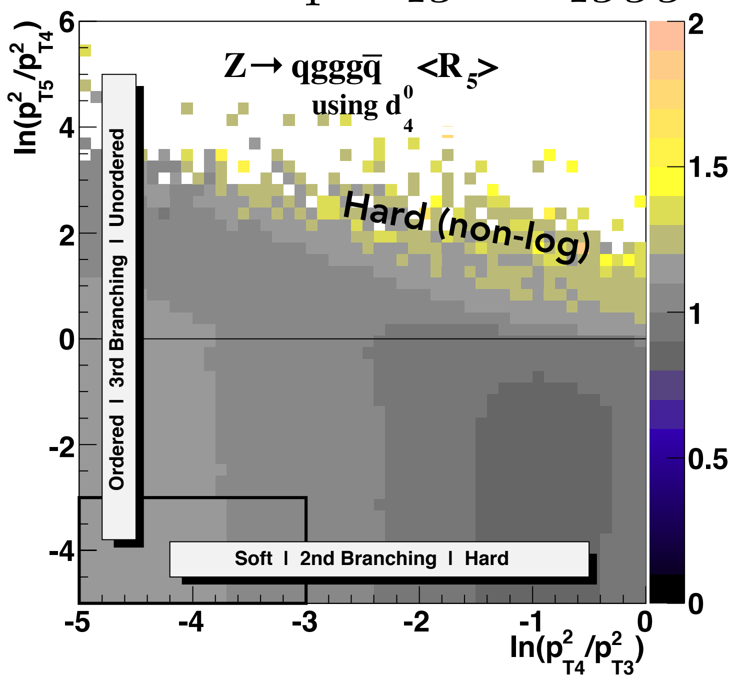
If the resulting path is **ordered**: populate by iterated 2→3 (with 2→4 MEC factors)
If **unordered**, keep clustering; direct 2→n

Clustering terminates when we reach a $Q_n > \min(p_{T2}, p_{T3}, \ldots)$
⇨ defines point as 2→2+m (NB: so far we only do 2→3 and 2→4!)

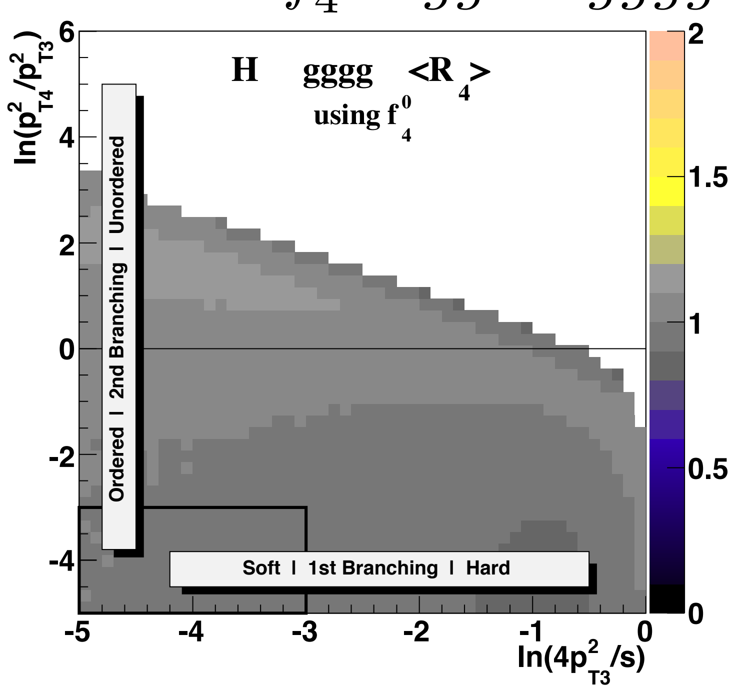
PHASE-SPACE DISTRIBUTIONS

Li & PZS: PLB771 (2017) 59

Test of d_4^0 : $qg \to qggg$



Test of f_4^0 : $gg \to gggg$



Actual shower runs:

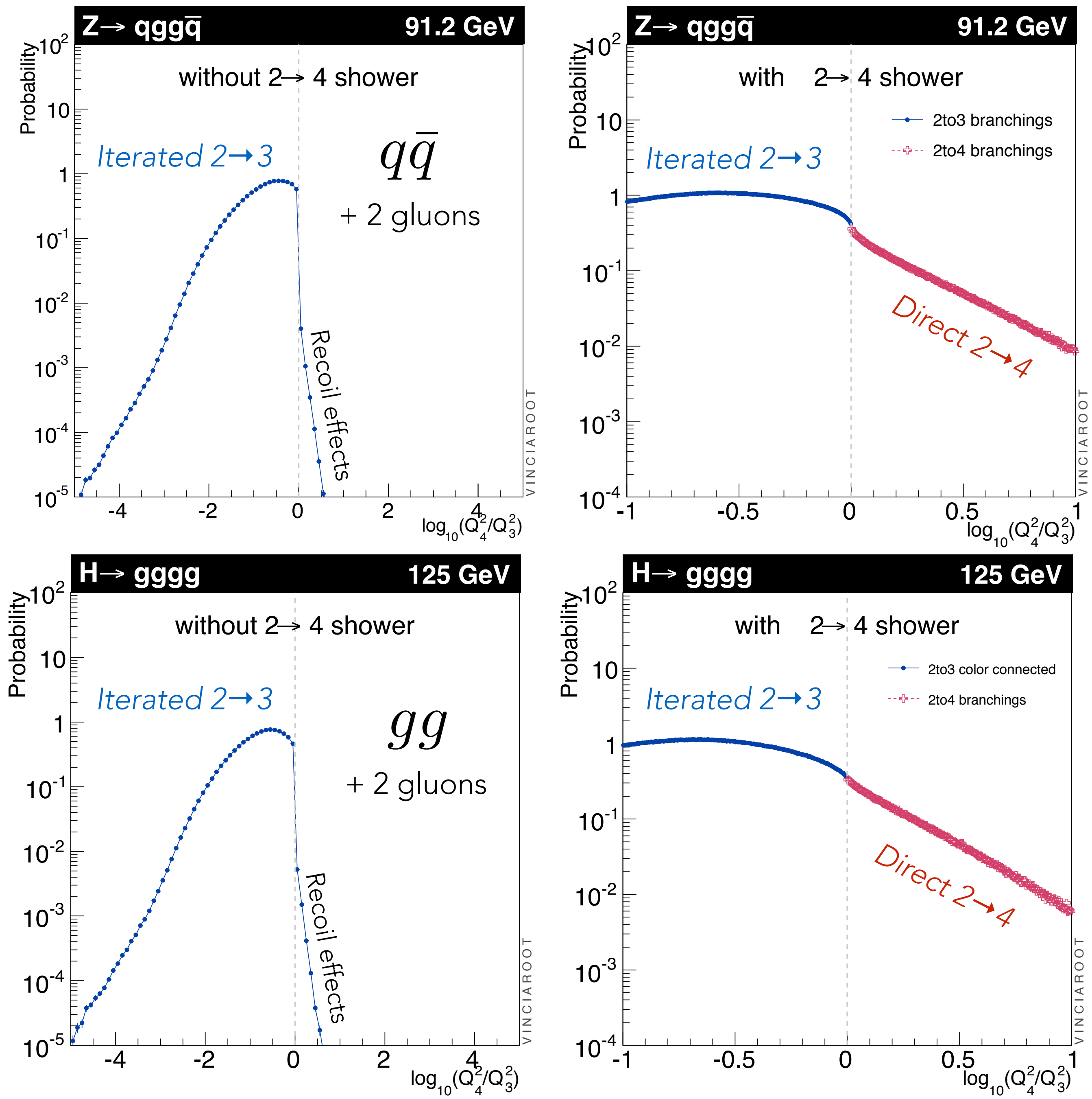


Figure 3: Top left: the ratio of sequential clustering scales Q_4/Q_3 for a strongly ordered $2 \to 3$ shower, for $Z \to qg\bar{q}$ (on log-log axes). Top right: closeup of the region around $Q_4/Q_3 \sim 1$, with 2→4 branchings included. Bottom row: the same for $H \to gggg$. .

Details of trial functions etc, see Li & PZS: PLB771 (2017) 59

VIRTUAL CORRECTIONS

Disclaimer

No established literature for antenna-evolved **fragmentation functions**

Known results (e.g., one-loop antenna functions) available mainly in context of F.O. subtraction terms, not diff. eqs. / exponentation / resummation

⇨ Had to (re)invent much of what follows as we went along

Clustering sequence = series of on-shell representations of the momentum flow; terminates when representation consistent with ordering (allowing to sum over unresolved degrees of freedom below that scale)

Sudakov factor for an antenna $\Delta(Q_0^2, Q^2) = \Delta_{2\to3}(Q_0^2, Q^2)\Delta_{2\to4}(Q_0^2, Q^2)$

Clustering corresponding to Q^2 is: Ordered Unordered (but next one up is ordered)

⇨ Starting from Q_0 (with inclusive sum over all unresolved 2→3 and 2→4 branchings below it), evolve to given Q: exclusive above Q, inclusive below

To define one-loop MEC: compare expansions of shower Sudakov factors to 2nd-order antenna functions

VIRTUAL MECS @ SECOND ORDER

Proof of concept case: second-order correction to q-qbar antenna emitting a gluon

Hartgring, Laenen, PZS: JHEP 1310 (2013) 127

A.k.a. $e^+e^- \to$ 3 jets at $O(\alpha_s^2)$

(at that time, we used smooth ordering for double-real; now direct $2\to4$)

Matching equation for one-loop virtual:

Exclusive 3-jet cross section above Q_4, for $Q_4 \to 0$ (in dim.reg.)

(Could stop at hadronisation scale → power corrections in Q_{had})

All-orders shower answer

$$\left|M_{Z\to q\bar{q}}\right|^2 A_3^0(Q^2)\left(1+V_3^{q\bar{q}}\right)\Delta_{2\to3}(Q_0^2,Q^2)\Delta_{3\to4}(Q^2,0) \overset{\mathcal{O}(\alpha_s^2)}{\to} |M_3^0|^2\left(1+\frac{2\mathrm{Re}\left[M_3^0 M_3^{1*}\right]}{|M_3^0|^2}\right)$$

Normalisation (best cross section available) (consistent with defining unit probability via unitarity at given order)

$2\to3$ antenna function (tree-level, with MEC) with α_s evaluated at μ_{PS}

One-loop matching term, to be solved for

$2\to3$ Sudakov factor : no branchings from starting scale Q_0 to resolution scale of 3-parton configuration, Q

$3\to4$ Sudakov factor : no branchings from scale of 3-parton configuration, Q to zero (in dim. reg.)

Fixed-Order $O(\alpha_s^2)$

(in dim. reg.)

(renormalised at $\mu = \mu_{ME}$)

Hartgring, Laenen, PZS, JHEP 1310 (2013) 127

Solve for V3

$$\left|M_{Z\to q\bar{q}}\right|^2 A_3^0(Q^2)\left(1+V_3^{q\bar{q}}\right)\Delta_{2\to 3}(Q_0^2,Q^2)\Delta_{3\to 4}(Q^2,0) \stackrel{\mathcal{O}(\alpha_s^2)}{\to} |M_3^0|^2\left(1+\frac{2\mathrm{Re}\left[M_3^0 M_3^{1*}\right]}{|M_3^0|^2}\right)$$

Non-divergent NLO correction
→ **positive-definite** NLO antenna

Poles ← Cancel if Q is IR safe → Poles

Partial cancellations
Use to define LL evolution so as to have no (resummable) logs left

Double Logs
Single Logs
(β-dependent logs)

+ transcendentality-0

Can do some Sudakov integrals analytically

But not all → split into analytic and numerical parts

Use that smooth-ordering already gave a good approximation, which can be integrated fairly easily

E.g.: $$\Delta_{3\to 4} = 1 - \sum_{a\in 1,2}\int_{\mathrm{ord}} \mathrm{d}\Phi_{\mathrm{ant}}\, a_{3\to 4}\frac{a_{2\to 4}}{a_{2\to 3}a_{3\to 4}+a'_{2\to 3}a'_{3\to 4}} + \mathcal{O}(\alpha_s^2)$$

ordering boundary — complicated 2→4 ME-correction factor

$$\pm\sum_{a\in 1,2}\int \mathrm{d}\Phi_{\mathrm{ant}}\, a_{3\to 4} P_{\mathrm{imp}}$$

Doable analytically;
contains all single-unresolved poles

Difference done numerically;
(slow but can be parametrised in terms of two invariants)

Hartgring, Laenen, PZS, JHEP 1310 (2013) 127

Solve for V_3

$$\left|M_{Z\to q\bar{q}}\right|^2 A_3^0(Q^2)\left(1+V_3^{q\bar{q}}\right)\Delta_{2\to3}(Q_0^2,Q^2)\Delta_{3\to4}(Q^2,0) \overset{\mathcal{O}(\alpha_s^2)}{\rightarrow} |M_3^0|^2\left(1+\frac{2\mathrm{Re}\left[M_3^0 M_3^{1*}\right]}{|M_3^0|^2}\right)$$

Non-divergent NLO correction
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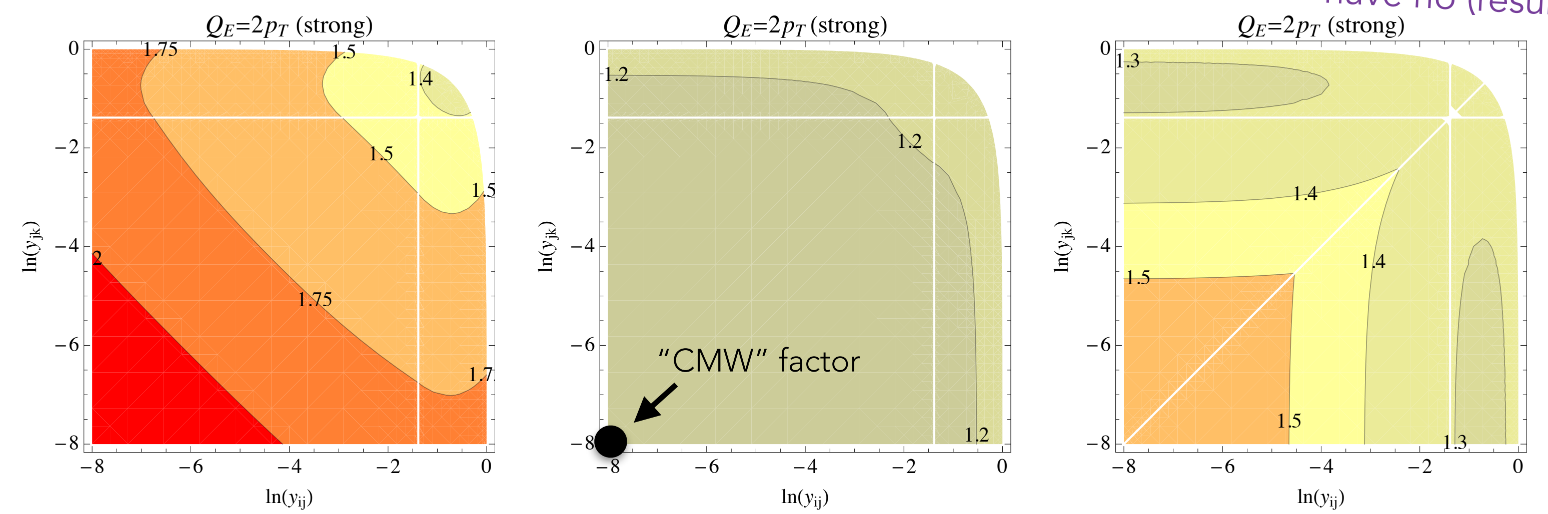
Use to define LL evolution so as to have no (resummable) logs left → Double Logs, Single Logs, (β-dependent logs)

+ transcendentality-0

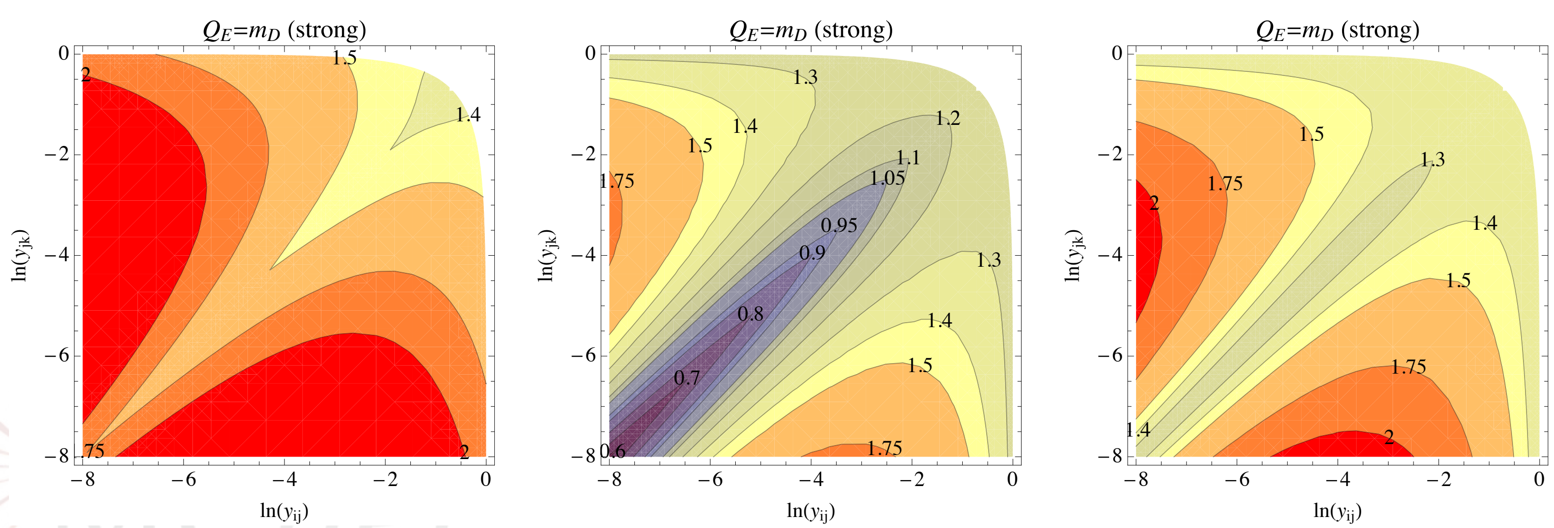


(a) $\mu_{PS}=\sqrt{s}$ (b) $\mu_{PS}=p_\perp$ (c) $\mu_{PS}=m_D$

Evolution-Variable dependence:

$$Q_E = p_\perp = \frac{m_{12}m_{23}}{m_{123}}$$

$$Q_E = m_D = \min(m_{12},m_{23})$$



(a) $\mu_{PS}=\sqrt{s}$ (b) $\mu_{PS}=p_\perp$ (c) $\mu_{PS}=m_D$

(β-dependent shower term):

$$a_{g/q\bar{q}}\Big|_{\mu_R=\mu_{PS}} \rightarrow$$

$$\left(1+\frac{\alpha_s}{2\pi}\frac{11N_C-2n_F}{6}\ln\left(\frac{\mu_{\mathrm{ME}}^2}{\mu_{\mathrm{PS}}^2}\right)+\mathcal{O}(\alpha_s^2)\right)\ a_{g/q\bar{q}}\Big|_{\mu_R=\mu_{\mathrm{ME}}}$$

ADDING QG AND GG PARENTS

with direct 2→4 instead of smooth ordering

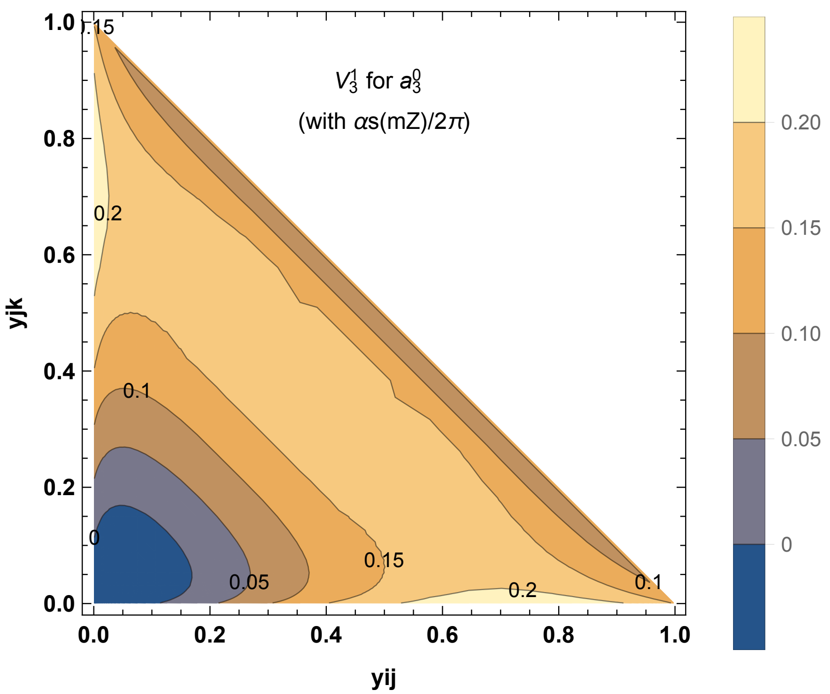
Work in progress…
Plots by Hai Tao Li

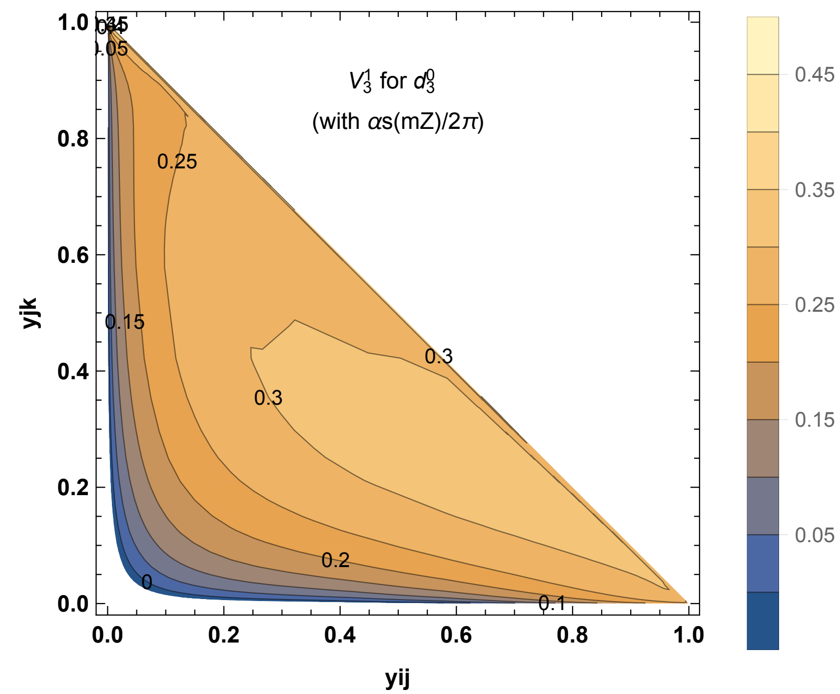
Differential NLO K-Factors for 2→3 kernels

$Q\bar{Q} \to QG\bar{Q}$

(new treatment)

From Z decay



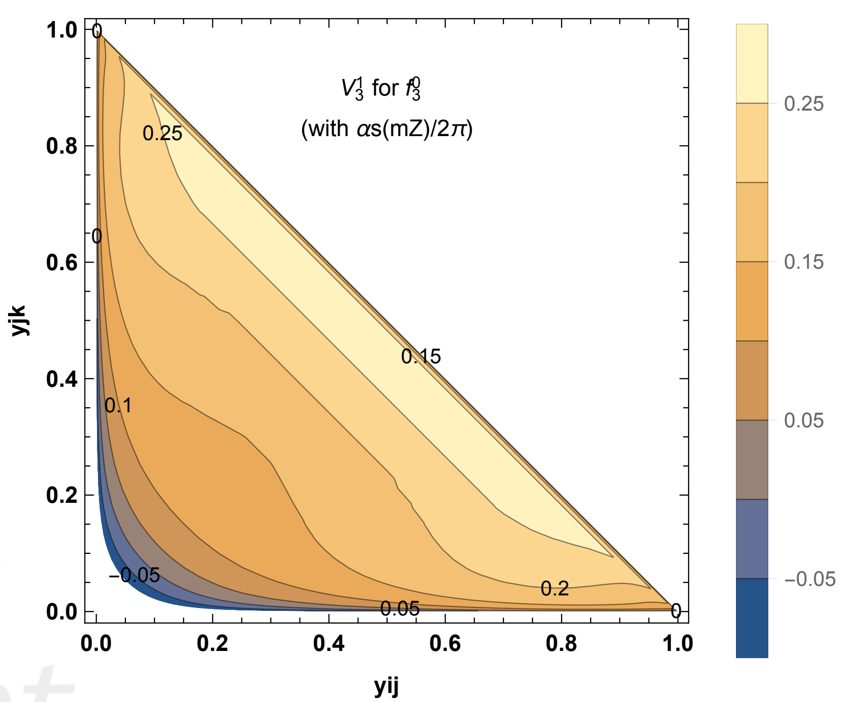


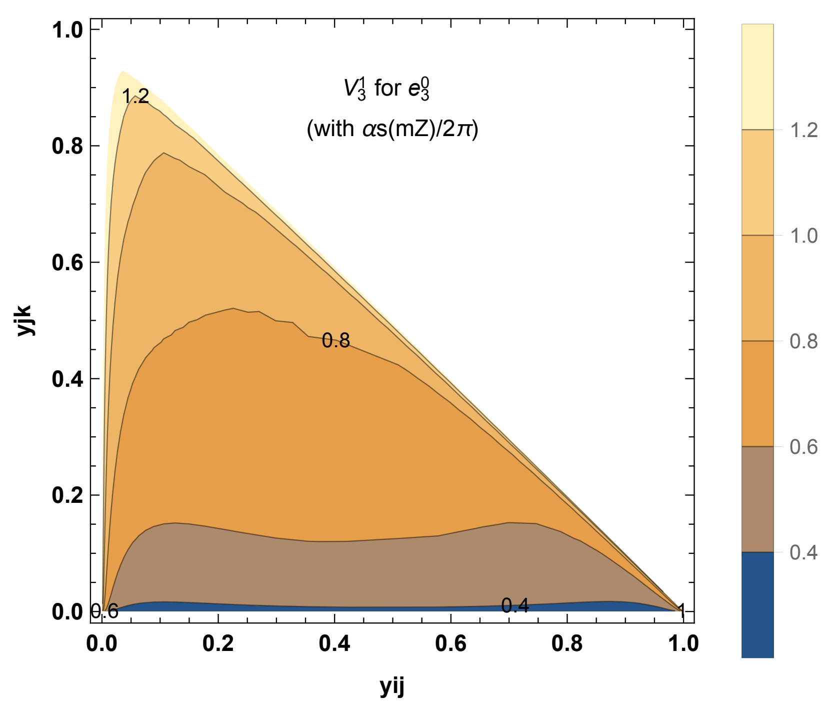
$QG \to QGG$

From χ decay

$GG \to GGG$

From H decay





$QG \to Q\bar{Q}'Q'$

Note: large corrections for g→qq

(leading pole only $1/y_{jk}$)

Li & PZS: PLB771 (2017) 59

Putting 2→3 and 2→4 together ⇨ evolution equation for dipole-antenna at O(α_s^2):

$$\frac{d\Delta(Q_0^2, Q^2)}{dQ^2} = \int d\Phi_{\rm ant} \left[\delta(Q^2 - Q^2(\Phi_3))\, a_3^0 \right.$$

$$\times \left(1 + \frac{a_3^1}{a_3^0} + \sum_{s\in a,b} \int_{\rm ord} d\Phi^s_{\rm ant}\, R_{2\to 4}\, s_3' \right) \Delta(Q_0^2, Q^2)$$

$$\left. + \sum_{s\in a,b} \int_{\rm unord} d\Phi^s_{\rm ant} \delta(Q^2 - Q^2(\Phi_4)) R_{2\to 4}\, s_3\, s_3' \Delta(Q_0^2, Q^2) \right]$$

Iterated 2→3 with (finite) one-loop correction (- → 0)

Direct 2→4 (as sum over "a" and "b" subpaths)

(2→)3→4 antenna function

(2→)3→4 MEC

2→4 as explicit product x MEC

Only generates double-unresolved singularities, not single-unresolved

Note: the equation is formally identical to:

$$\frac{d}{dQ^2}\Delta(Q_0^2, Q^2) =$$
$$\int \frac{d\Phi_3}{d\Phi_2}\, \delta(Q^2 - Q^2(\Phi_3)) \left(a_3^0 + a_3^1\right) \Delta(Q_0^2, Q^2)$$
$$+ \int \frac{d\Phi_4}{d\Phi_2}\, \delta(Q^2 - Q^2(\Phi_4))\, a_4^0\, \Delta(Q_0^2, Q^2)\,, \quad (3)$$

But on this form, the pole cancellation happens *between* the two integrals

FURTHER DETAILS & OUTLOOK

Further details

Since antenna functions are defined from specific physical matrix elements (GGG used Z, H, and χ decays), the corrections effectively include nonsingular terms for those processes

- Will probably use variations to estimate effects → uncertainty bands
- MECs (or merging) at given order could be used to cancel them

VINCIA 2 used a mixed evolution, with gluon emissions ordered in p_T and g→qq splittings ordered in m_{qq}

- Large log corrections at NLO → reverting to single evolution measure

When can others play with it?

Old NLO qq → qqg corrections already implemented in VINCIA 1

New paradigm; currently writing up longer paper with details and preparing for new code release, with Hai Tao Li. Expect (at least) a month or two.

Note: still only for (massless) final-state evolution.

- Big projects in their own right: ISR and radiation in resonance decays (+QED)

SUMMARY

GGG: full set of 2nd-order antenna functions (summed over colours and permutations)

Gehrmann-de Ridder, Gehrmann, Glover JHEP 0509 (2005) 056

We use: $Q\bar{Q} : N_C^2 A_3^1,\ N_C n_F \hat{A}_3^1,\ N_C^2 A_4,\ N_C n_F B_4,$

$QG : N_C^2 D_3^1,\ N_C n_F \hat{D}_3^1,\ n_f N_C E_3^1, n_f^2 \hat{E}_3^1,\ N_C^2 D_4,\ N_C n_F E_4$

$GG : N_C^2 F_3^1,\ N_C n_F \hat{F}_3^1, n_F N_C G_3^1,\ n_F^2 \hat{G}_3^1,\ N_C^2 F_4,\ N_C n_F G_4,\ n_F^2 H_4$

(NB: GGG also provide subleading-colour and -flavour antenna functions; so far ignored)

(Colour-ordered *sub-antenna* functions defined using 2→3 as partitioning functions)

Direct 2→4 branchings interleaved with iterated ME-corrected 2→3 ones

2→4 resolved in $Q_4 = \min(p_{T4a}, p_{T4b})$

Based on eikonal × P_{imp} × eikonal integrated over intermediate (unphysical) Q_3 scale

Two trial channels, one for each path; overlap with iterated 2→3 removed by vetos

Iterated 2→3 resolved in Q_3 (as before, with veto if colour neighbour has lower scale and is unordered), with differential second-order (NLO) "K-factor" correction

(For default shower parameters, the NLO correction is well-controlled; can become large if using "wrong" renormalisation scales, evolution measures, etc.)

All radiation functions positive-definite

2→4 from tree-level positivity (and no overlap); 2→3 since written explicitly as [1 + $O(\alpha_s)$]

New evolution paradigm beyond LL looking promising so far.

Backup Slides

2→4 TRIAL GENERATION

$$\frac{1}{(16\pi^2)^2} a^{2\to4}_{\rm trial} = \frac{2}{(16\pi^2)^2} a^{2\to3}_{\rm trial}(Q_3^2) P_{\rm imp} a^{2\to3}_{\rm trial}(Q_4^2)$$
$$= C \left(\frac{\alpha_s}{4\pi}\right)^2 \frac{128}{(Q_3^2+Q_4^2)Q_4^2} \,. \qquad (15)$$

In particular, the trial function for sector A (B) is independent of momentum p_6 (p_3) which makes it easy to translate the $2 \to 4$ phase spaces defined in eq. (6) to shower variables. Technically, we generate these phase spaces by oversampling, vetoing configurations which do not fall in the appropriate sector.

Accept ratio:

$$P^{2\to4}_{\rm trial} = \frac{\alpha_s^2}{\hat{\alpha}_s^2} \frac{a_4}{a^{2\to4}_{\rm trial}}$$

Solution for constant trial α_s

$$\mathcal{A}^{\rm trial}_{2\to4}(Q_0^2, Q^2) = C\, I_\zeta \frac{\ln(2)\hat{\alpha}_s^2}{8\pi^2} \ln\frac{Q_0^2}{Q^2} \ln\frac{m^4}{Q_0^2 Q^2}$$

$$\Rightarrow \quad Q^2 = m^2 \exp\left(-\sqrt{\ln^2(Q_0^2/m^2) + 2f_R/\hat{\alpha}_s^2}\right)$$

where $f_R = -4\pi^2 \ln R/(\ln(2) C I_\zeta)$. (Same $I_{\rm zeta}$ as in GKS)

Solution for first-order running α_s (also used as overestimate for 2-loop running):

$$Q^2 = \frac{4\Lambda^2}{k_\mu^2}\left(\frac{k_\mu^2 m^2}{4\Lambda^2}\right)^{-1/W_{-1}(-y)} . \qquad (20)$$

Lambert W

where

$$y = \frac{\ln k_\mu^2 m^2/4\Lambda^2}{\ln k_\mu^2 Q_0^2/4\Lambda^2} \exp\left[-f_R b_0^2 - \frac{\ln k_\mu^2 m^2/4\Lambda^2}{\ln k_\mu^2 Q_0^2/4\Lambda^2}\right] ,$$