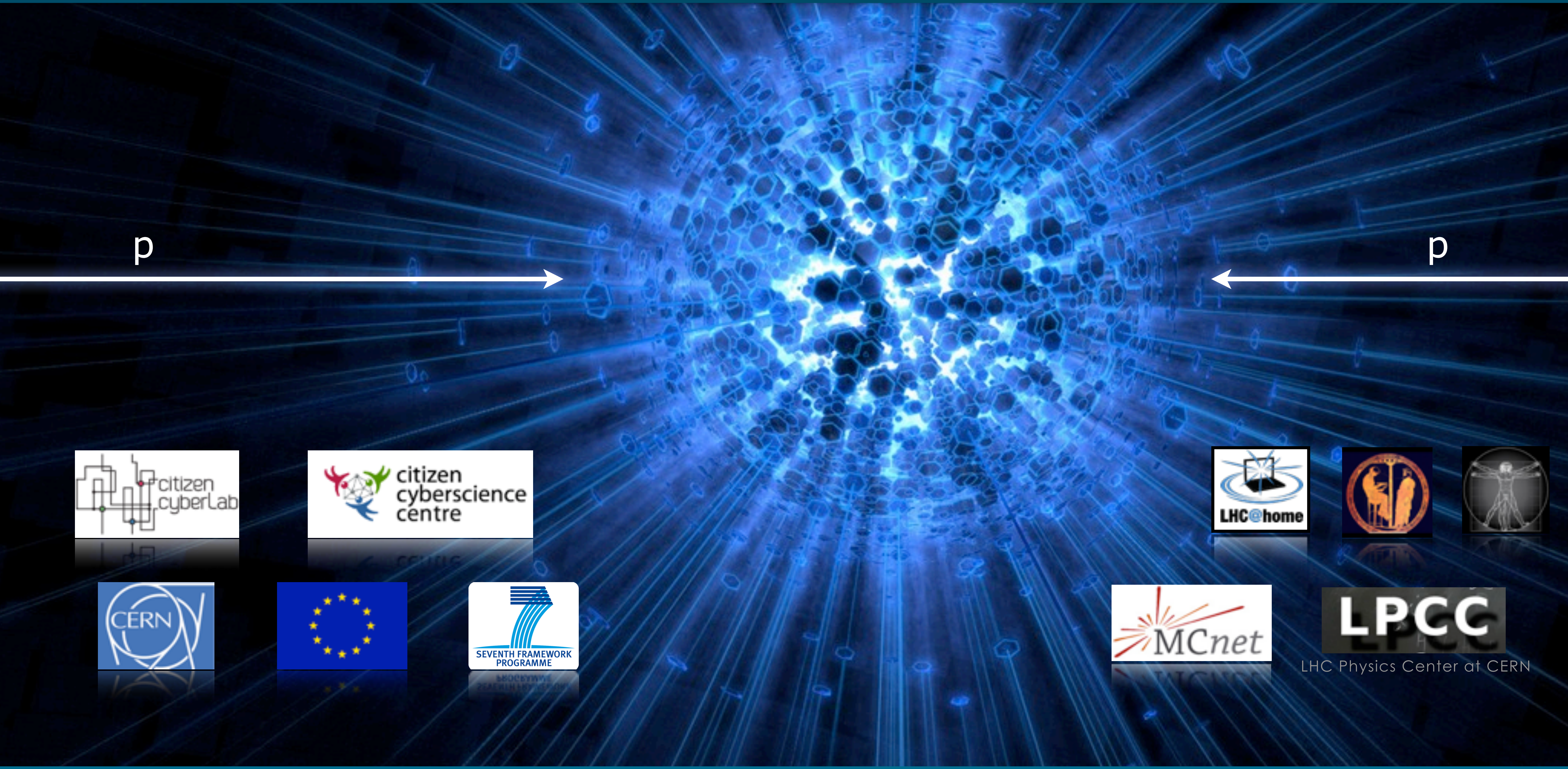
Event Generator Physics

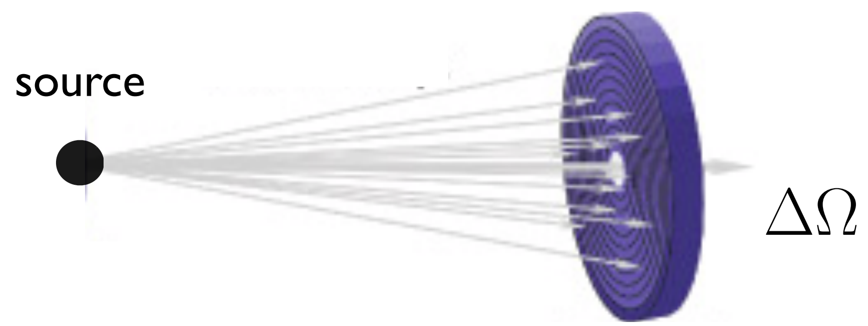
Peter Skands (CERN Theoretical Physics Dept)



Terascale Monte Carlo School
DESY, Hamburg - March 2014

Lectures 1+2: QCD & MC

Scattering Experiments



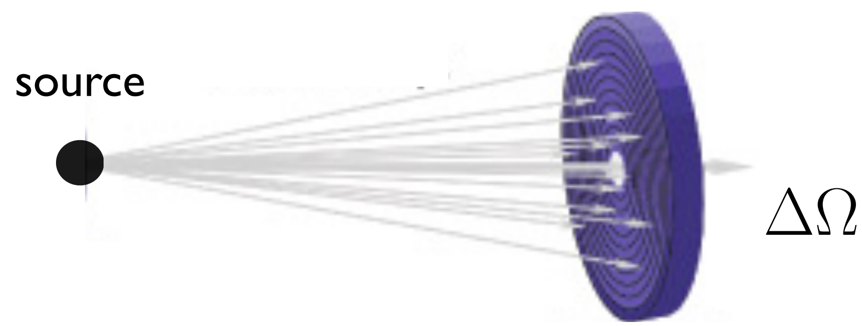
LHC detector
Cosmic-Ray detector
Neutrino detector
X-ray telescope
…

→ Integrate differential cross sections over specific phase-space regions

Predicted number of counts = integral over solid angle

$$N_{\mathrm{count}}(\Delta\Omega) \propto \int_{\Delta\Omega} \mathrm{d}\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$$

Scattering Experiments



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In particle physics:
Integrate over all quantum histories
(+ interferences)

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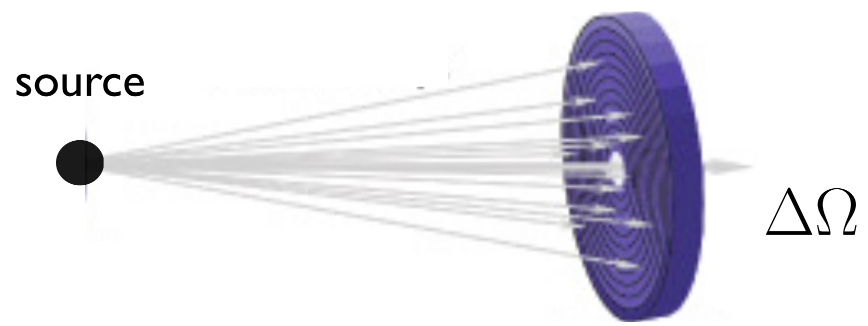

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Complicated integrands?
→ Numerical approaches
High-dimensional phase spaces?
→ Monte Carlo

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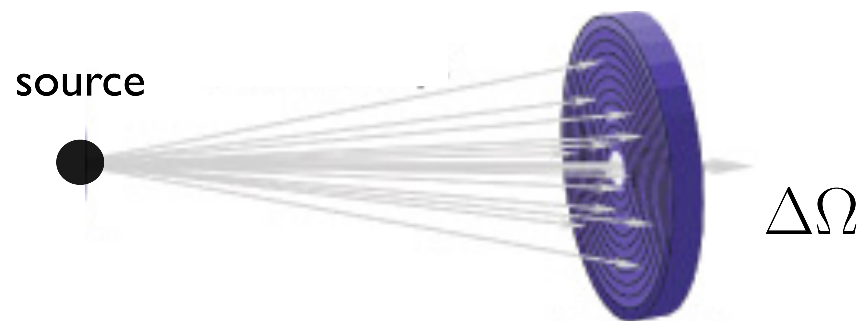

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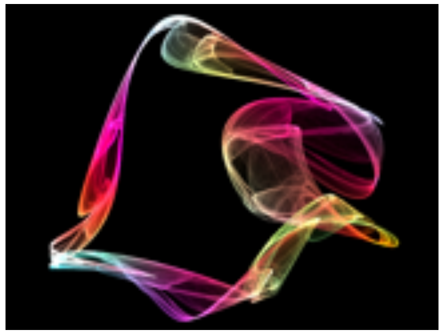
QCD Recap

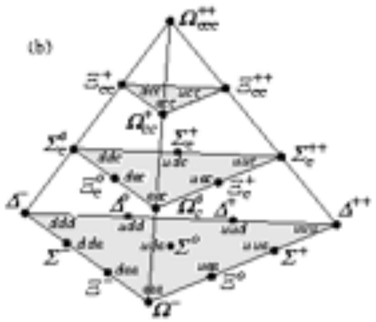
More than just a perturbative expansion in α_s

Emergent phenomena

Jets (the QCD fractal) ⟷ amplitude structures (in phase space) ⟷ fundamental quantum field theory. Precision jet (structure) studies.



Strings (strong gluon fields) ⟷ quantum-classical correspondence. String physics. Dynamics of hadronization phase transition.



Hadrons (incl excited states) ⟷ Spectroscopy, lattice QCD, (rare) decays, mixing, exotic states (e.g Ω_{ccc}, hadron molecules, …), light nuclei

$$\mathcal{L} = \bar{\psi}_q^i (i\gamma^\mu)(D_\mu)_{ij}\psi_q^j - m_q\bar{\psi}_q^i\psi_{qi} - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$$

Quark fields

$$\psi_q^j = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

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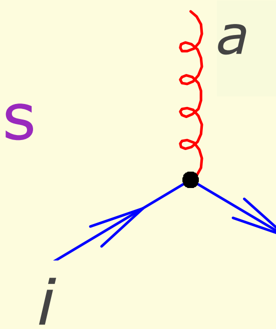
Covariant Derivative

$$D_{\mu ij} = \delta_{ij}\partial_\mu - \underline{i g_s T^a_{ij} A^a_\mu}$$

⇒ Feynman rules



a∈[1,8]
i,j∈[1,3]

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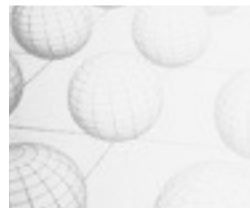

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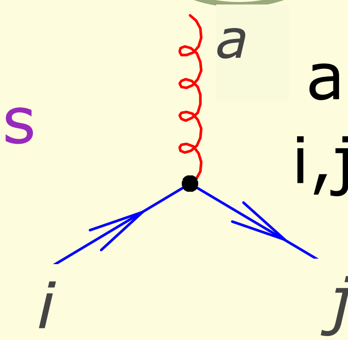

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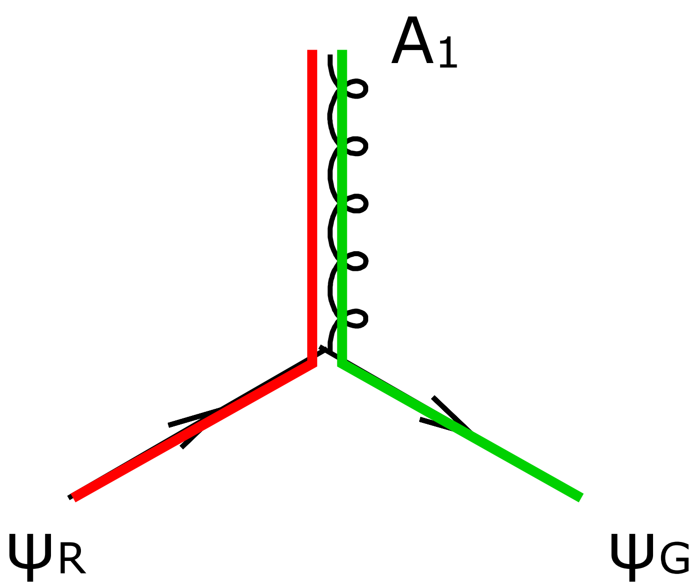
Gell-Mann Matrices *($T^a = ½\lambda^a$)*

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix}$$

Interactions in Color Space

Quark-Gluon interactions



$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

A_1 $\quad\quad$ ψ_R $\quad\quad$ ψ_G

Interactions in Color Space

Color Factors

All QCD processes have a “color factor”. It counts the enhancement from the sum over colors.

(or suppression if colors have to match)

~ how many “color paths” we can take

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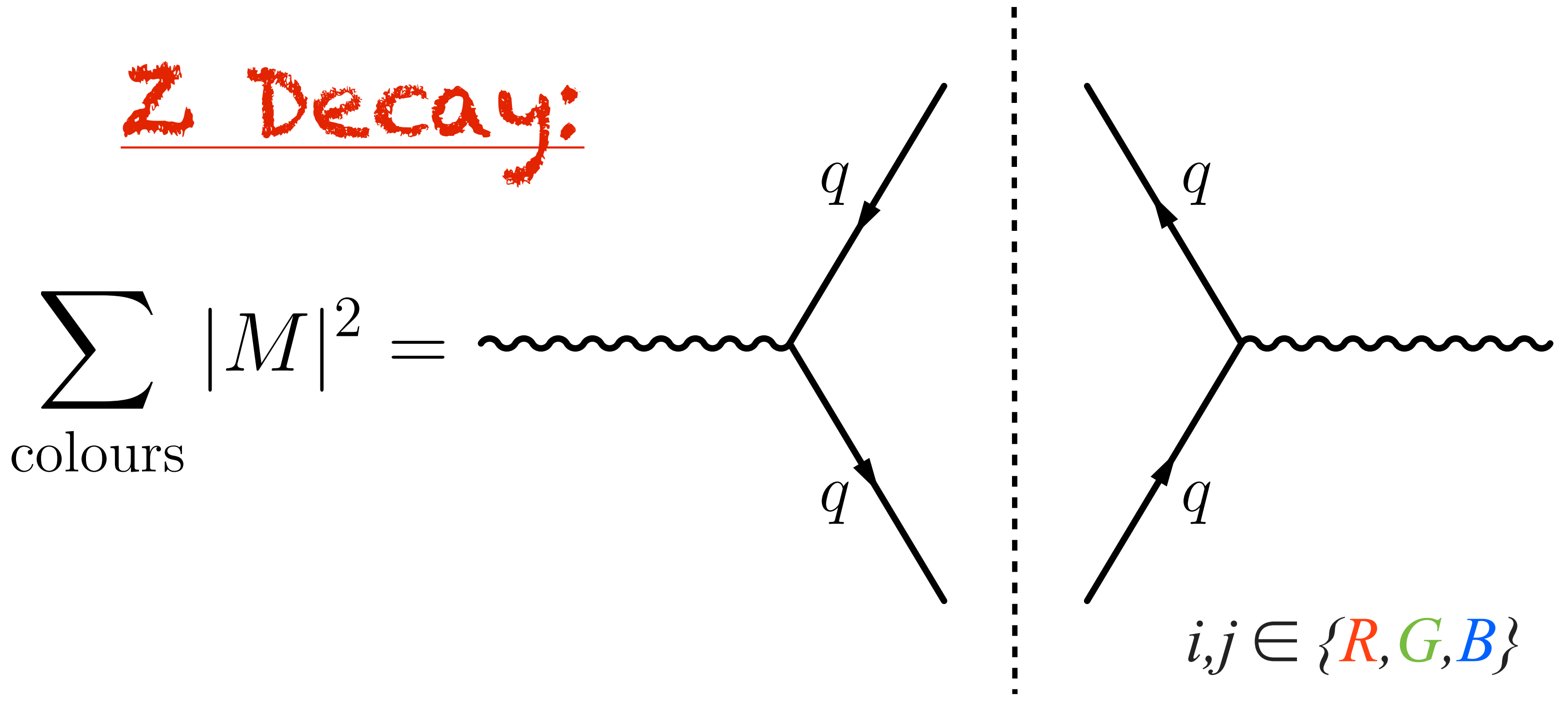

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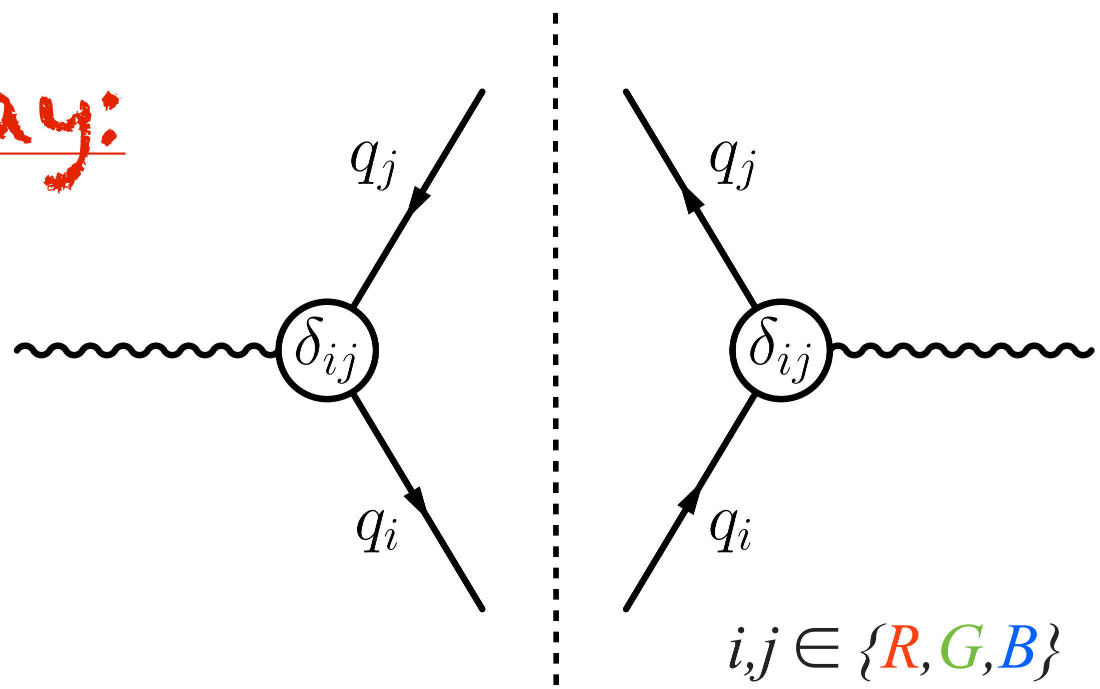
$$\sum_{\text{colours}} |M|^2 =$$



$$\propto \delta_{ij}\delta^*_{ji}$$

$$= \mathrm{Tr}[\delta_{ij}]$$

$$= N_C$$

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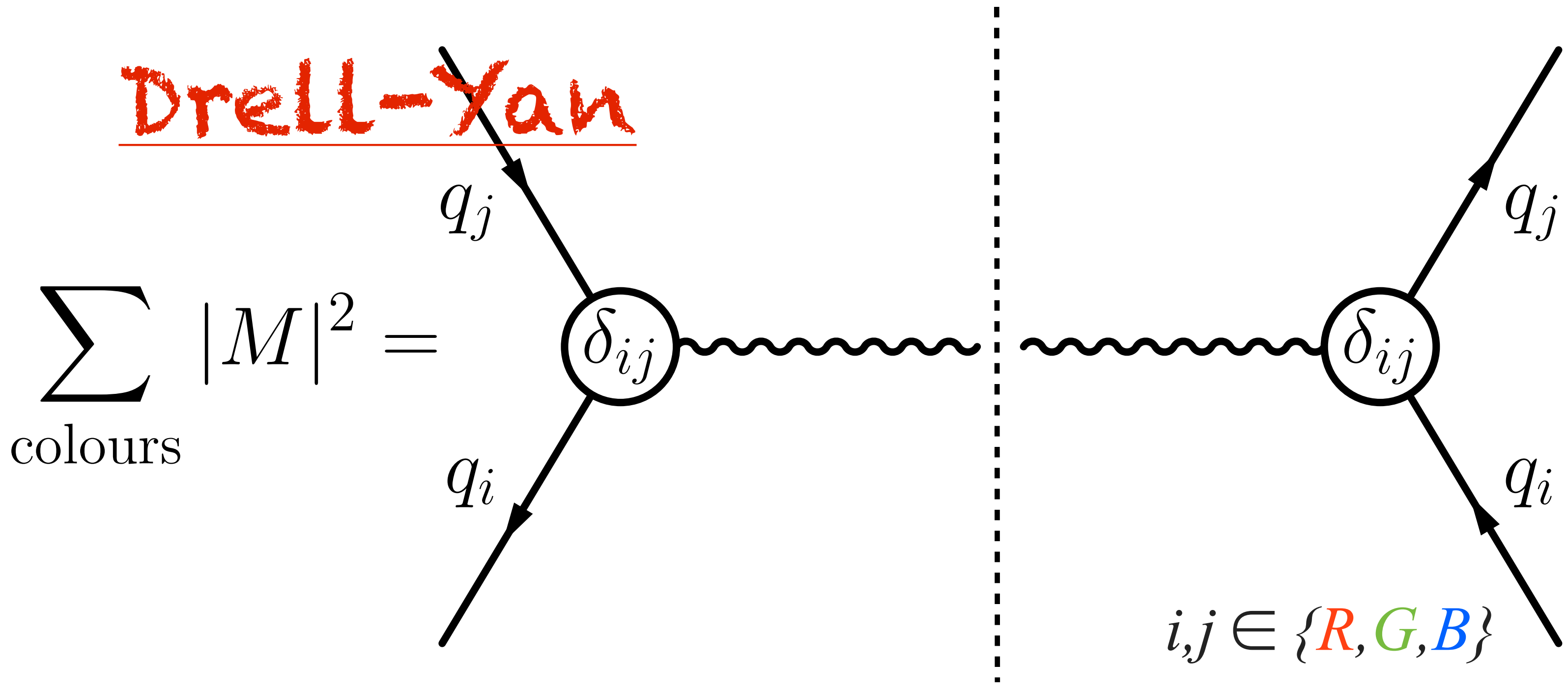

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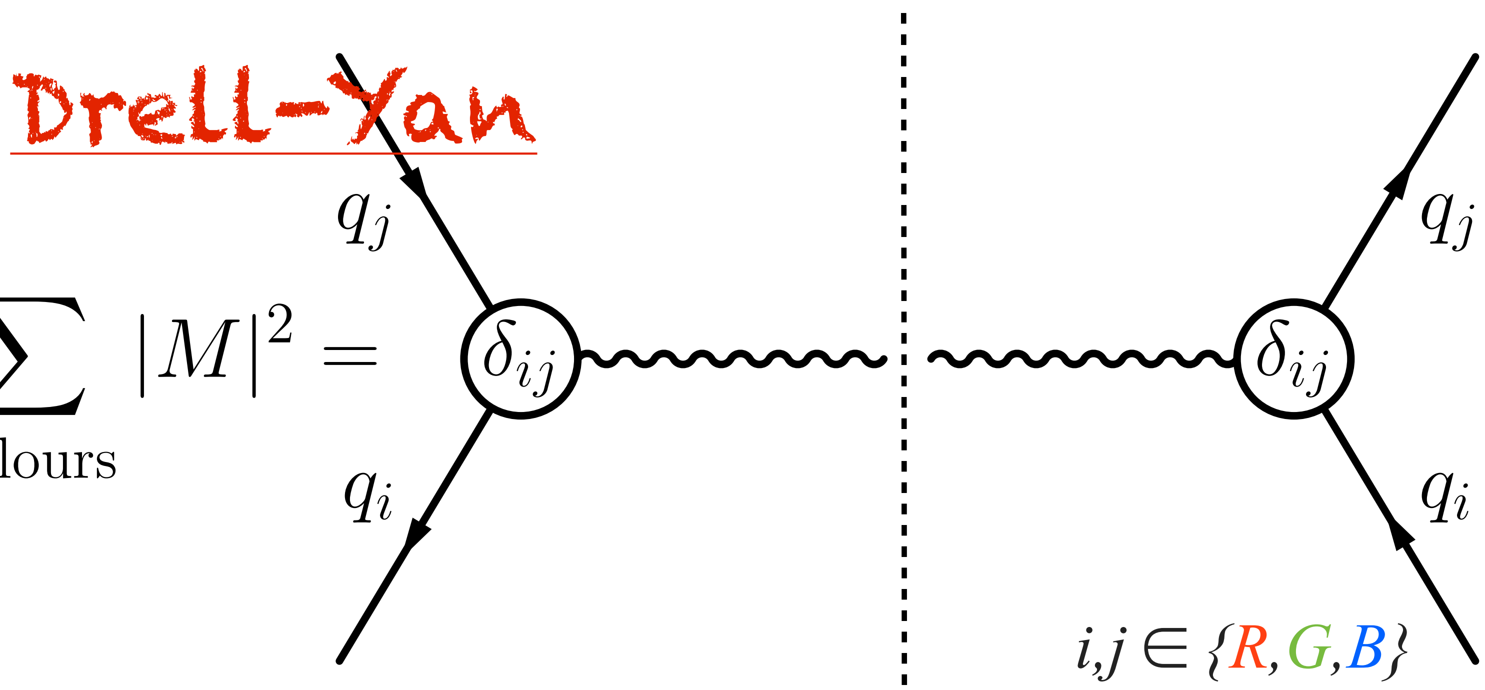
$$\frac{1}{9}\sum_{\mathrm{colours}} |M|^2 = \propto \delta_{ij}\delta^*_{ji} = \mathrm{Tr}[\delta_{ij}] = N_C$$



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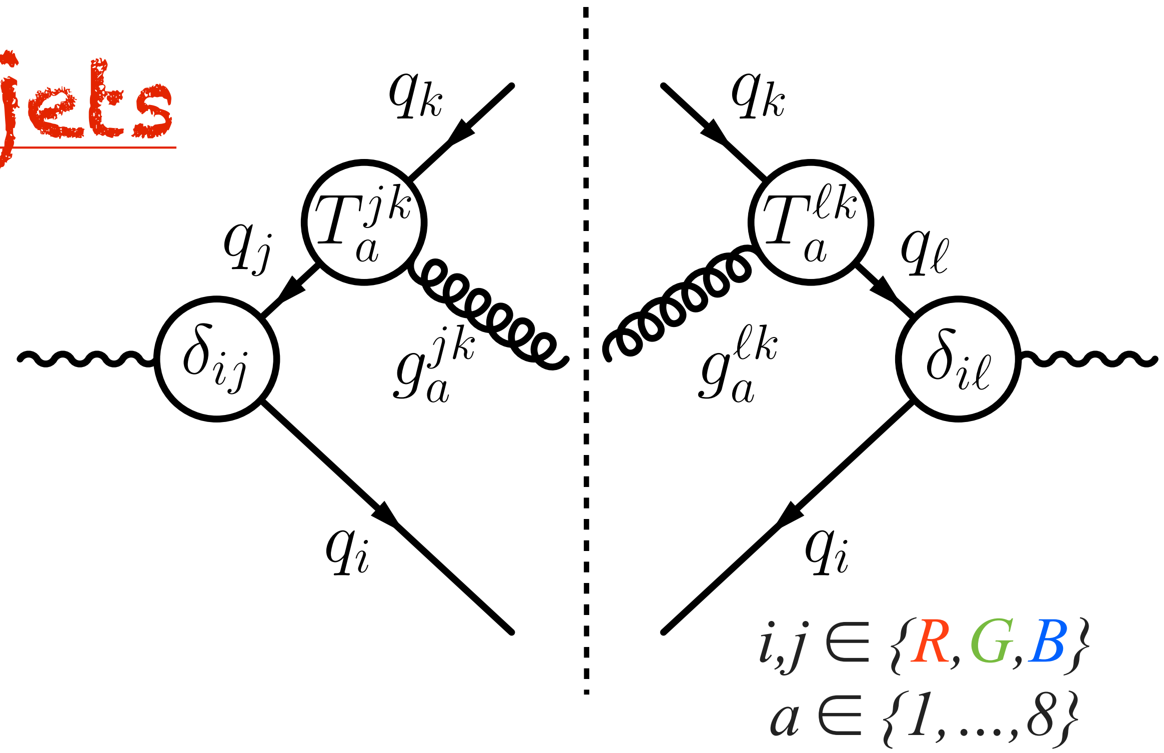
Z→3 jets

$$\sum_{\text{colours}} |M|^2 =$$



$$\propto \delta_{ij} T_a^{jk} (T_a^{\ell k} \delta_{i\ell})^*$$
$$= \mathrm{Tr}[T_a T_a]$$
$$= \frac{1}{2} \mathrm{Tr} \delta_{ab}$$
$$= 4$$

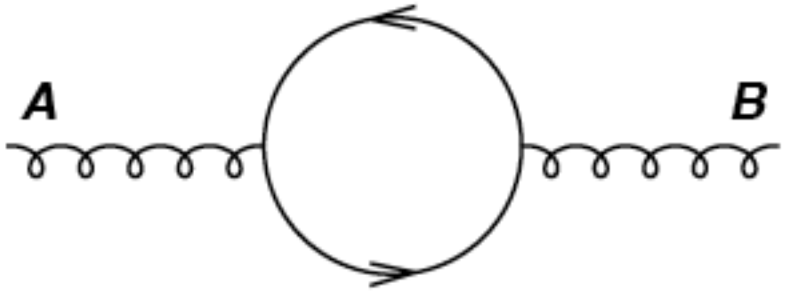
Quick Guide to Color Algebra

Color factors squared produce traces

Trace Relation

$$\mathrm{Tr}(t^A t^B) = T_R \delta^{AB}\,, \quad T_R = \frac{1}{2}$$

Example Diagram



(from ESHEP lectures by G. Salam)

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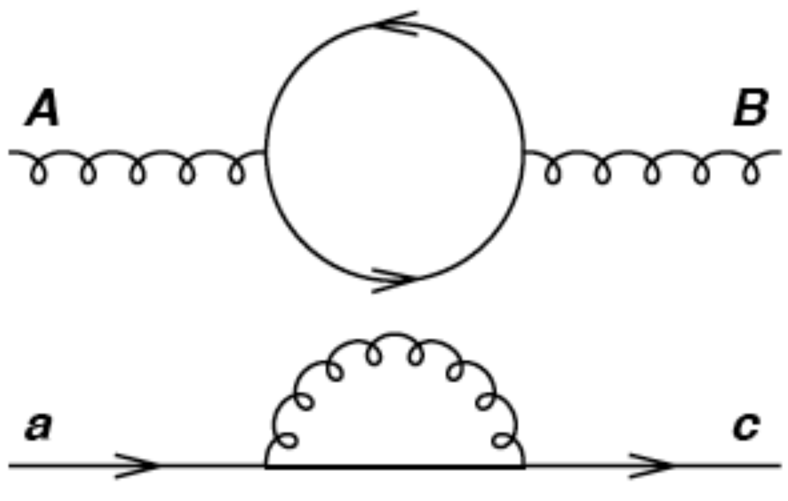

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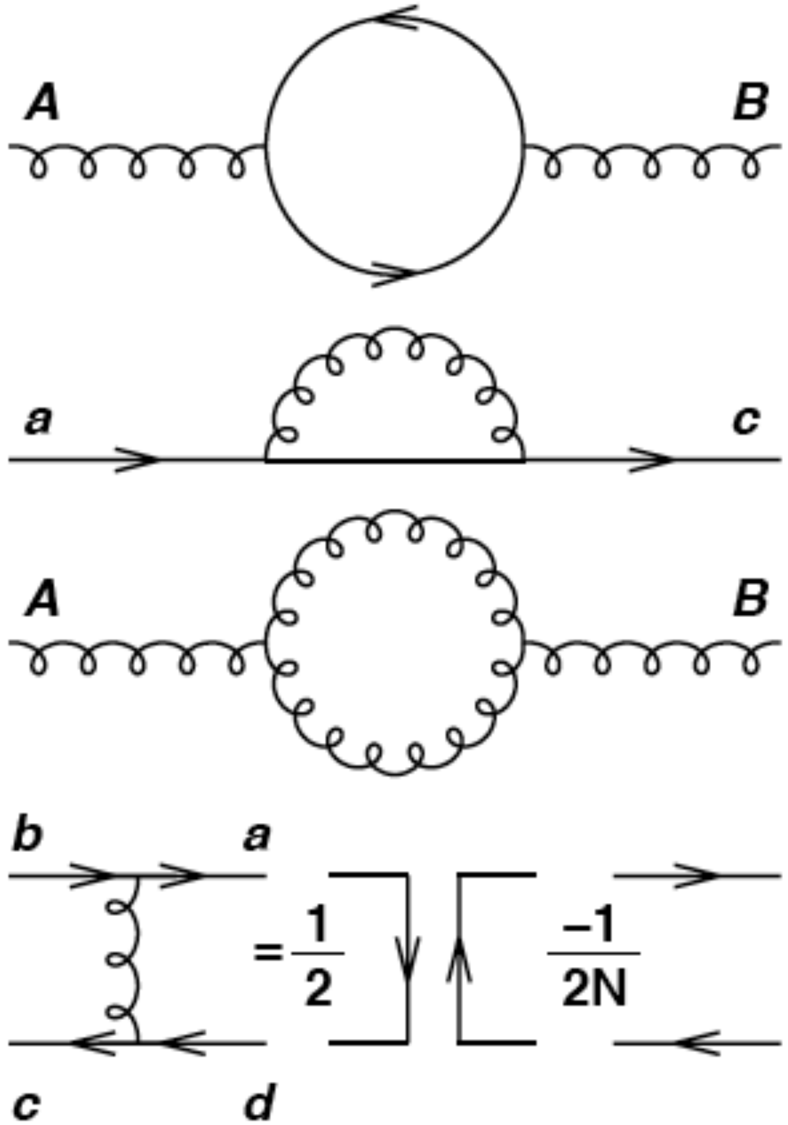
$$\sum_{C,D} f^{ACD} f^{BCD} = C_A \delta^{AB}\,, \quad C_A = N_c = 3$$

$$t^A_{ab} t^A_{cd} = \frac{1}{2}\delta_{bc}\delta_{ad} - \frac{1}{2N_c}\delta_{ab}\delta_{cd} \quad \text{(Fierz)}$$



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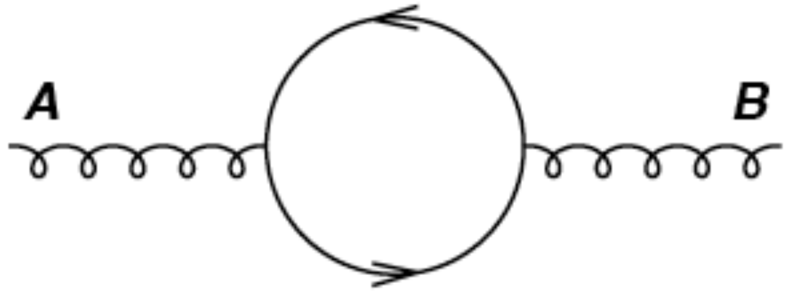
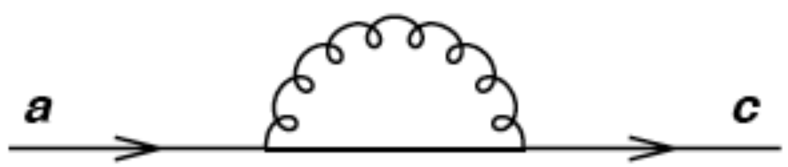
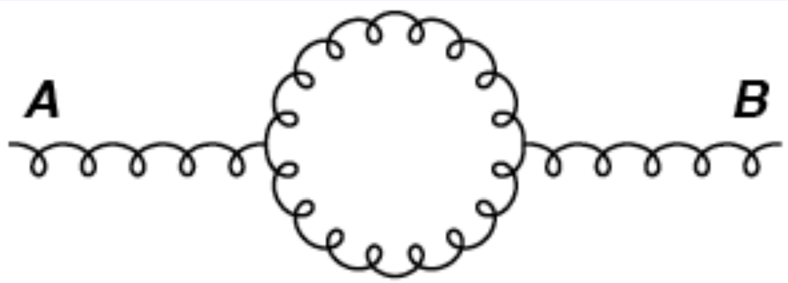
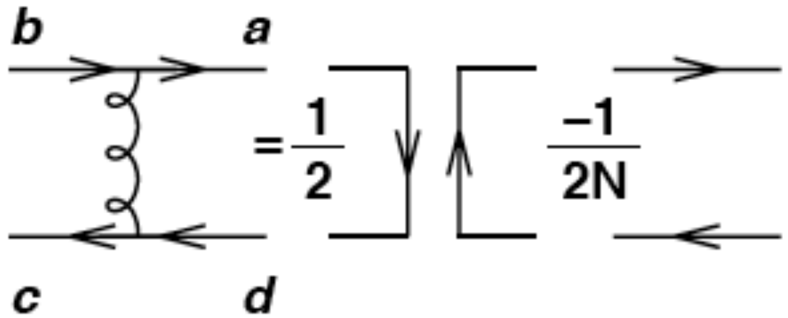
$\mathrm{Tr}(t^A t^B) = T_R \delta^{AB}\,, \quad T_R = \frac{1}{2}$	
$\sum_A t^A_{ab} t^A_{bc} = C_F \delta_{ac}\,, \quad C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}$	
$\sum_{C,D} f^{ACD} f^{BCD} = C_A \delta^{AB}\,, \quad C_A = N_c = 3$	
$t^A_{ab} t^A_{cd} = \frac{1}{2}\delta_{bc}\delta_{ad} - \frac{1}{2N_c}\delta_{ab}\delta_{cd}$ (Fierz)	

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The Gluon

Gluon-Gluon Interactions

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Gluon field strength tensor:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c$$

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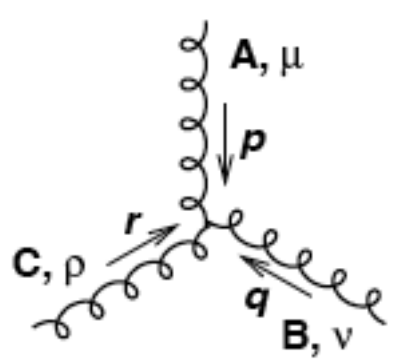
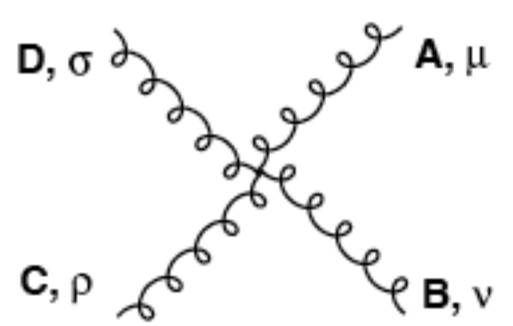


$$-g_s f^{ABC}[(p-q)^\rho g^{\mu\nu} + (q-r)^\mu g^{\nu\rho} + (r-p)^\nu g^{\rho\mu}]$$



$$-ig_s^2 f^{XAC} f^{XBD}[g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\gamma}] + (C,\gamma) \leftrightarrow (D,\rho) + (B,\nu) \leftrightarrow (C,\gamma)$$

Structure constants of SU(3):

$$f_{123} = 1$$

$$f_{147} = f_{246} = f_{257} = f_{345} = \frac{1}{2}$$

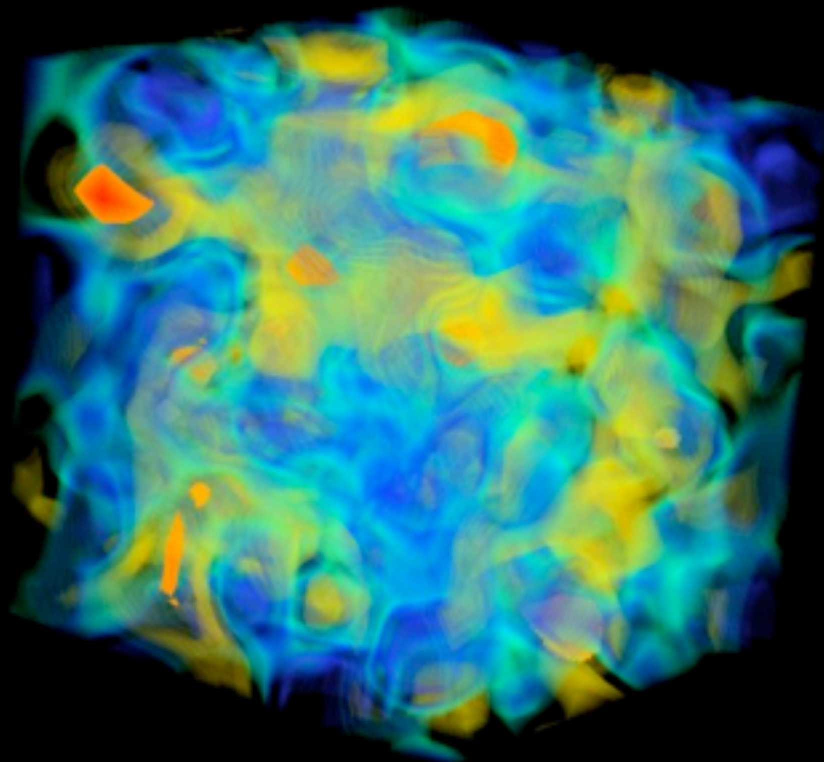
$$f_{156} = f_{367} = -\frac{1}{2}$$

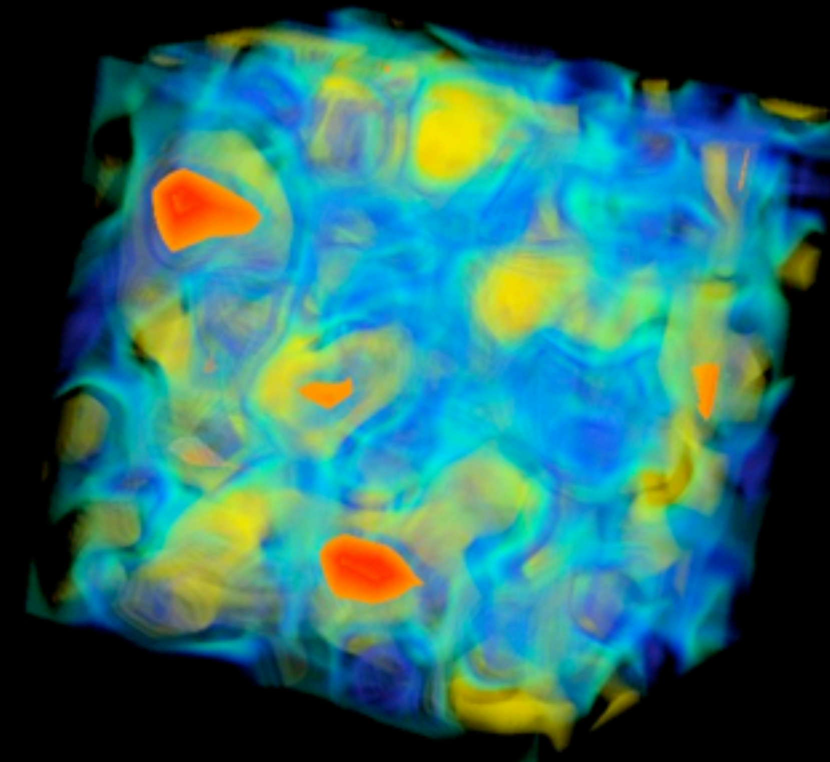
$$f_{458} = f_{678} = \frac{\sqrt{3}}{2}$$

Antisymmetric in all indices

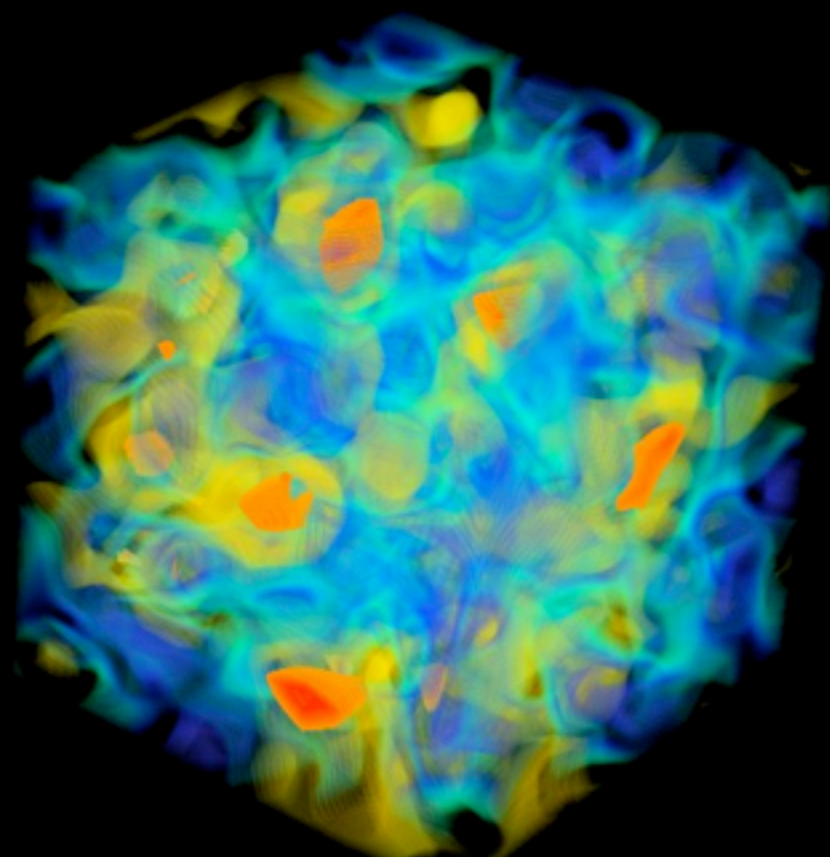
All other $f_{ijk} = 0$

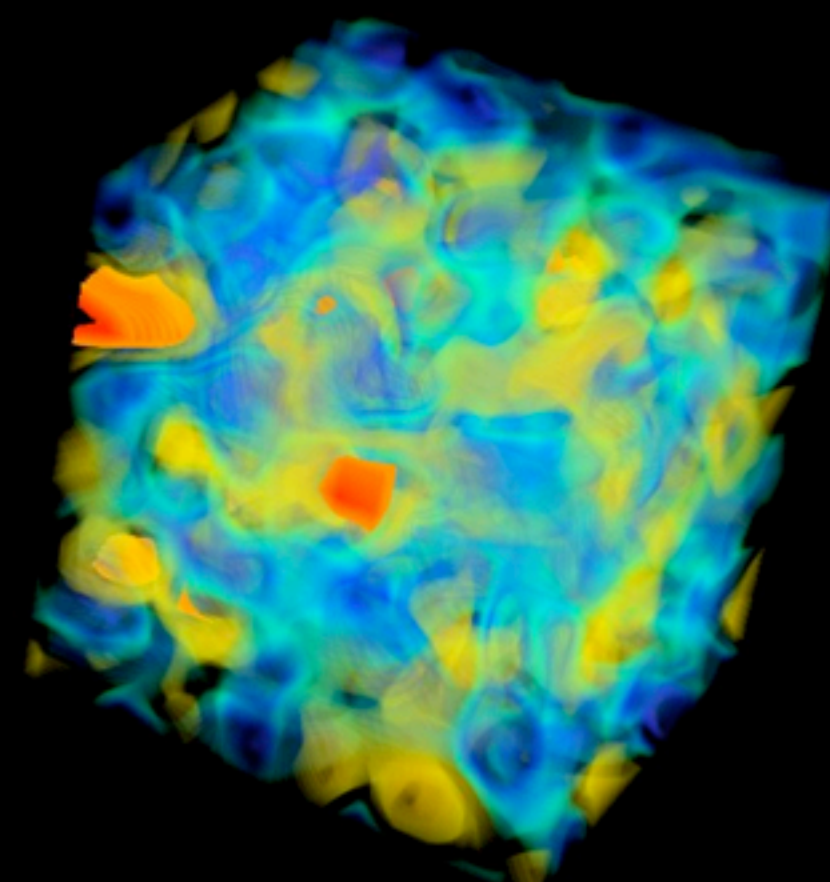
Vacuum Topological Charge, Data courtesy of M. McGuigan BNL-CSC, T. Izubuchi RIKEN-BNL, and S. Tomov University of Tennessee





QCD Dynamics





The Strong Coupling

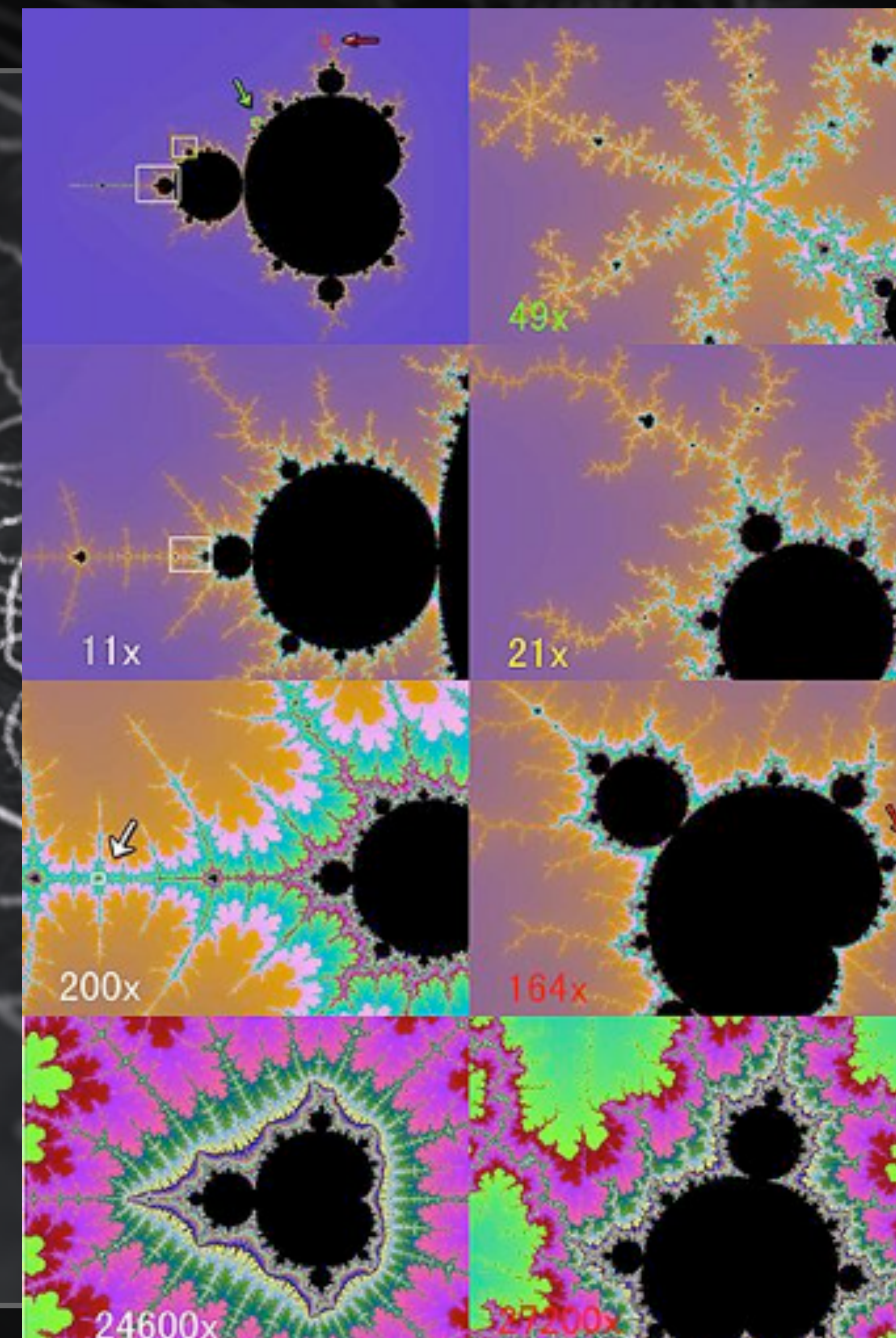
Bjorken scaling

To first approximation, QCD is SCALE INVARIANT (a.k.a. conformal)

A jet inside a jet inside a jet inside a jet …

If the strong coupling didn't "run", this would be absolutely true (e.g., N=4 Supersymmetric Yang-Mills)

As it is, α_s only runs slowly (logarithmically) → can still gain insight from fractal analogy



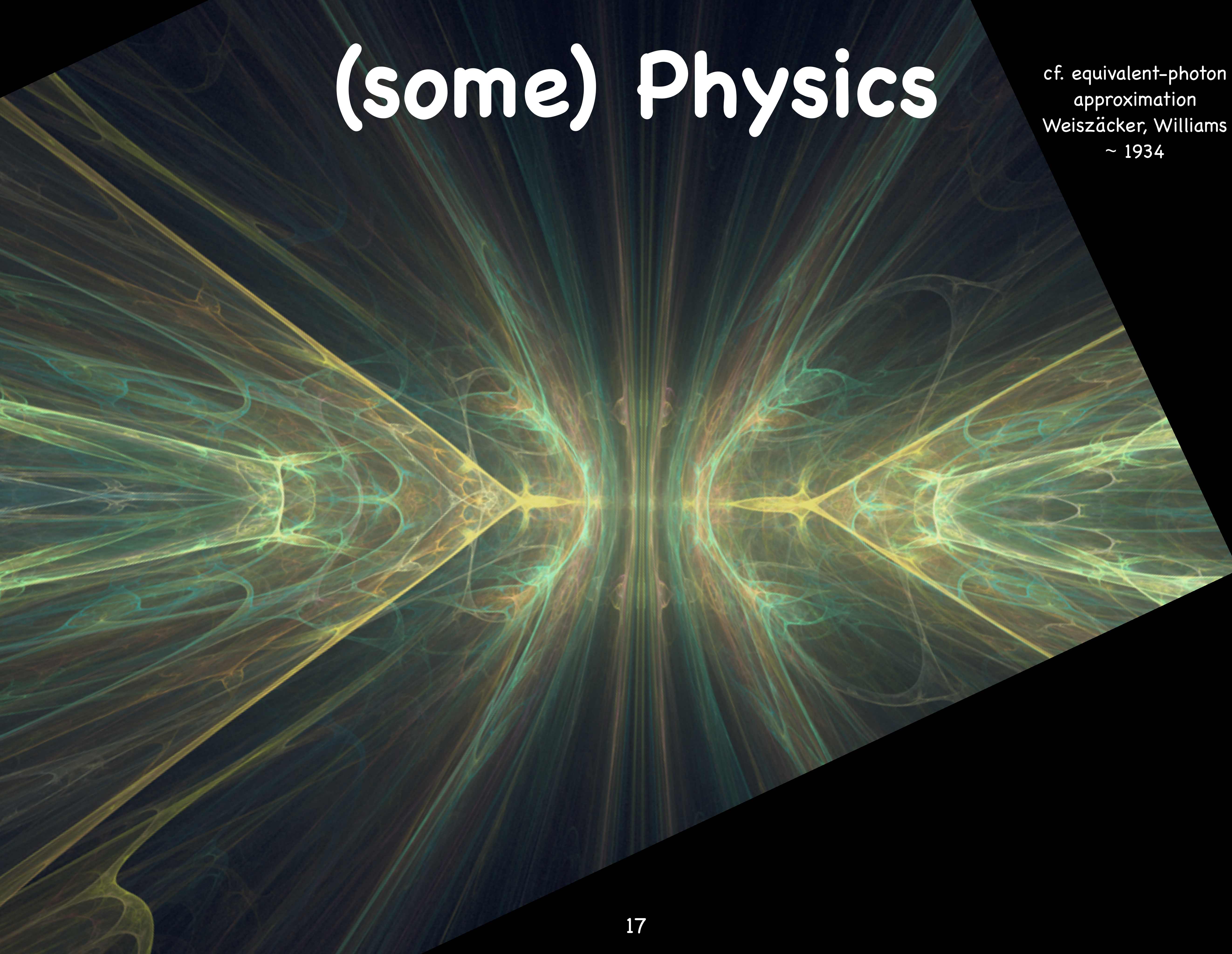
Note: I use the terms "conformal" and "scale invariant" interchangeably

Strictly speaking, conformal (angle-preserving) symmetry is more restrictive than just scale invariance

But examples of scale-invariant field theories that are not conformal are rare (eg 6D noncritical self-dual string theory)

(some) Physics

cf. equivalent-photon approximation
Weiszäcker, Williams
~ 1934



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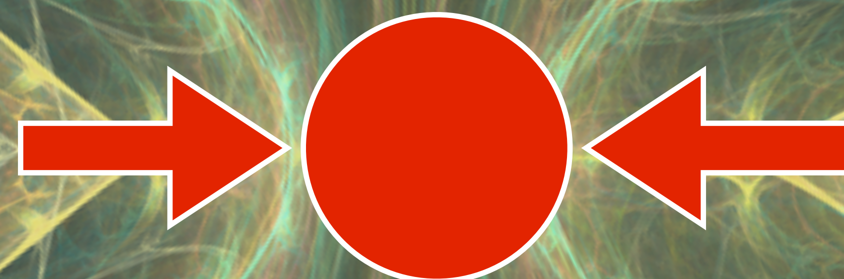
Charges Stopped or kicked



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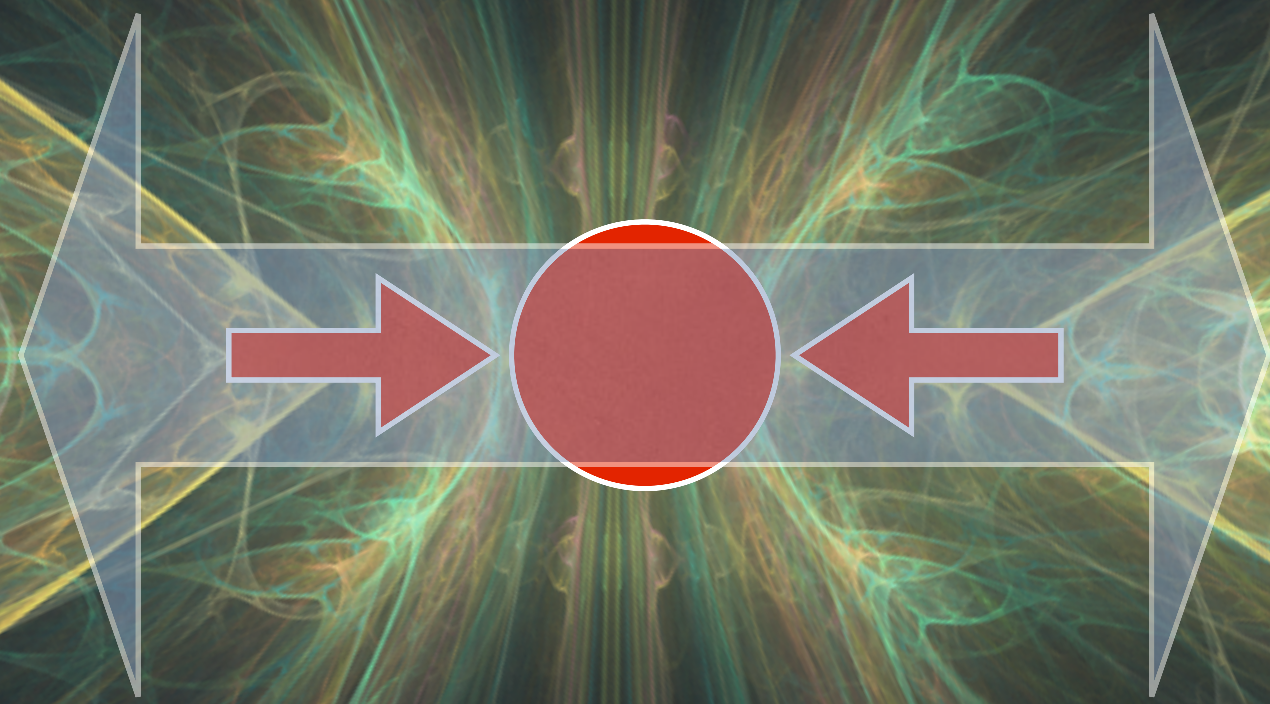

Associated field (fluctuations) continues

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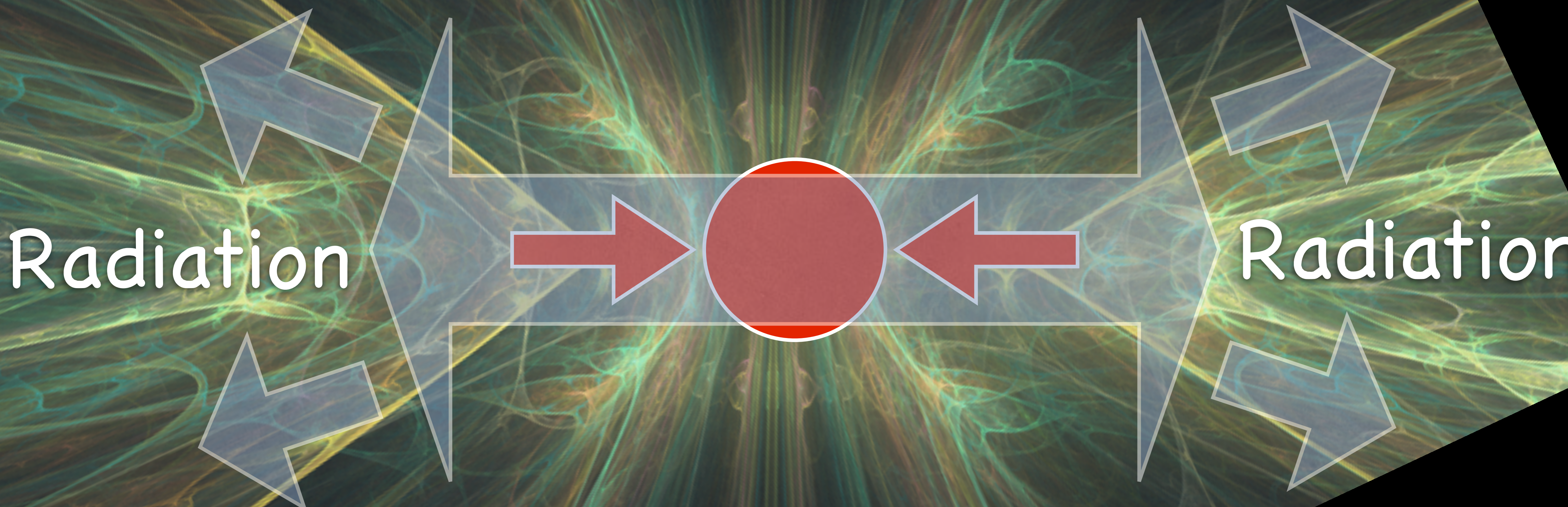

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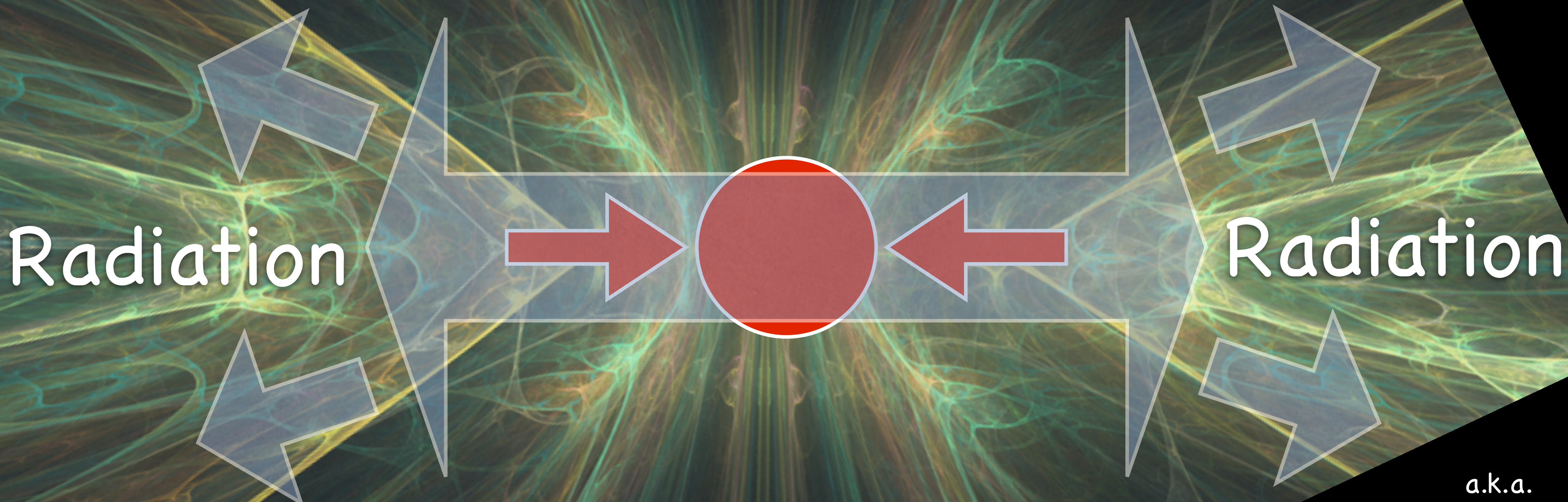
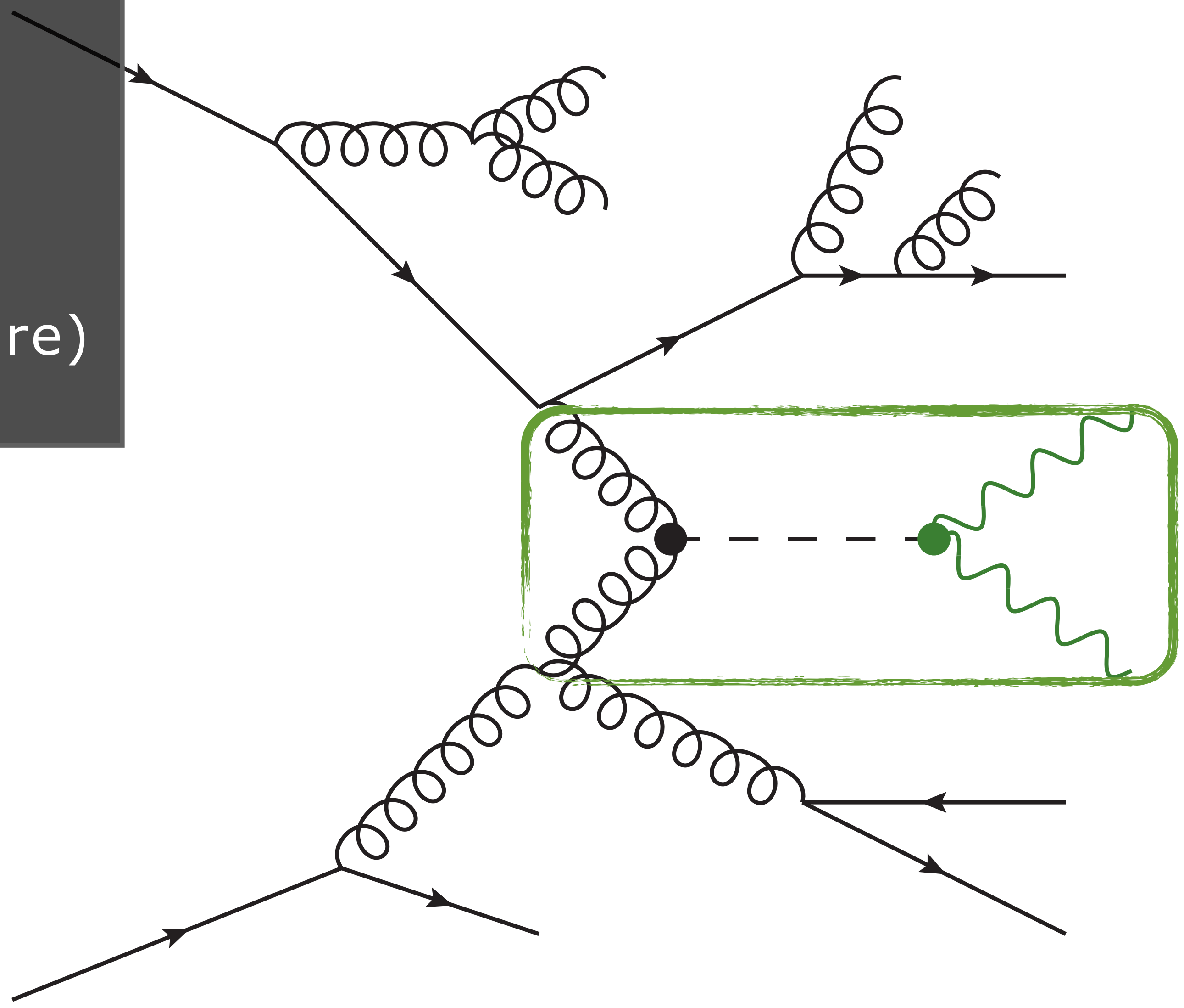


a.k.a.
Bremsstrahlung
Synchrotron Radiation

The harder they stop, the harder the fluctations that continue to become radiation

Jets ≈ Fractals

- **Most bremsstrahlung** is driven by divergent propagators → simple structure
- **Amplitudes factorize in singular limits** (→ universal "conformal" or "fractal" structure)



See: PS, *Introduction to QCD*, TASI 2012, arXiv:1207.2389

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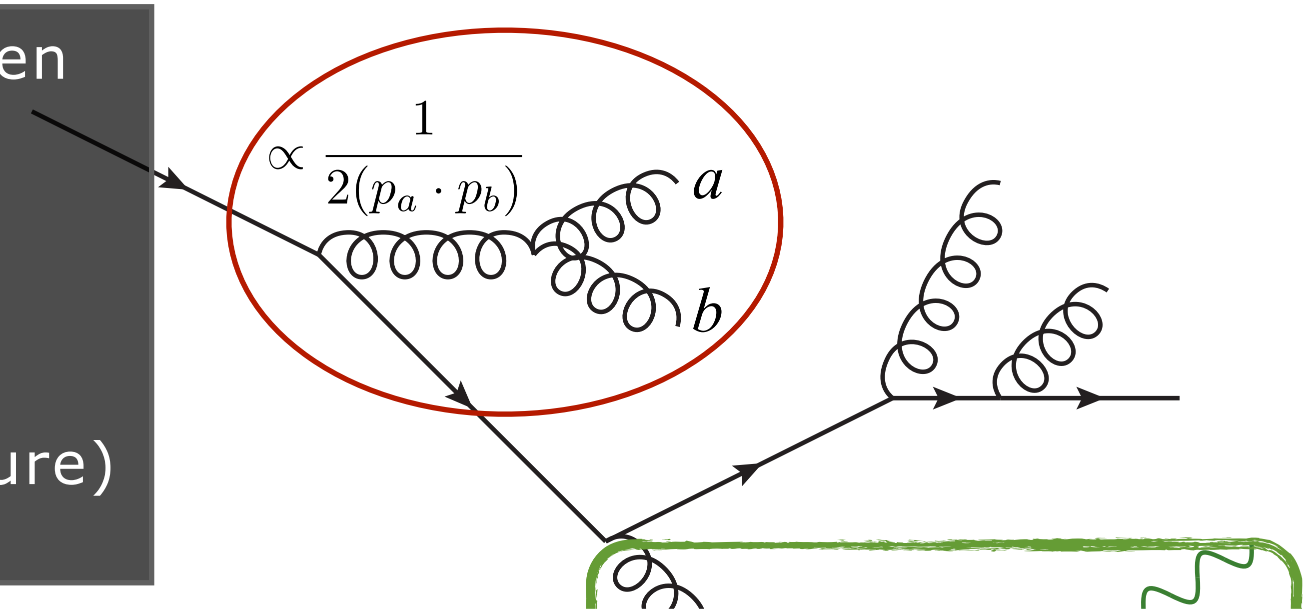

Partons ab → "collinear":

P(z) = DGLAP splitting kernels, with z = energy fraction = $E_a/(E_a+E_b)$

$$|\mathcal{M}_{F+1}(\ldots,a,b,\ldots)|^2 \stackrel{a||b}{\to} g_s^2 \mathcal{C} \frac{P(z)}{2(p_a \cdot p_b)} |\mathcal{M}_F(\ldots,a+b,\ldots)|^2$$

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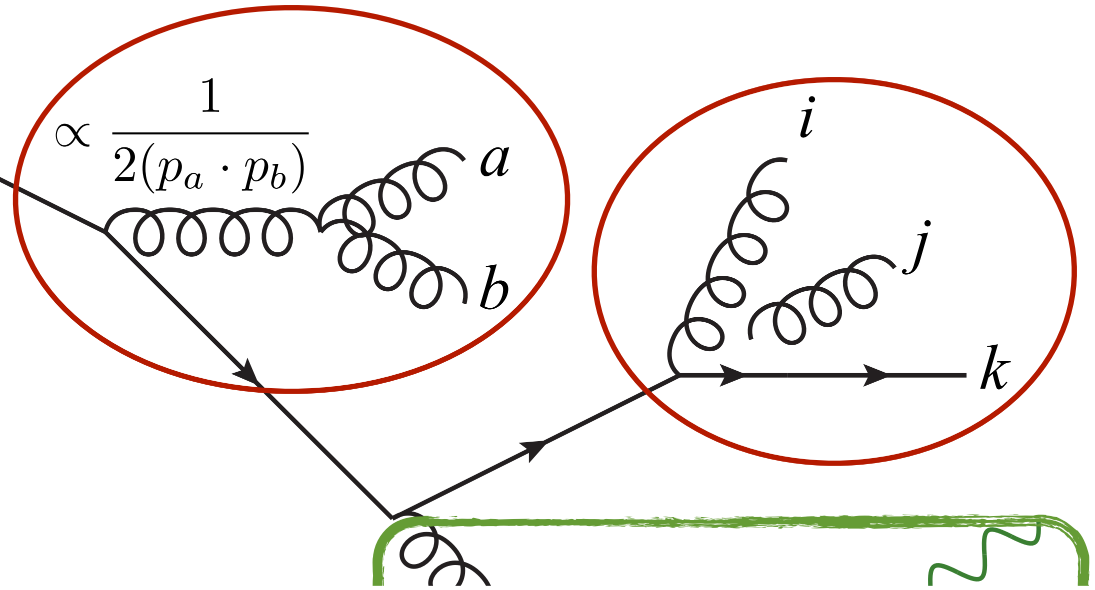

Partons ab → "collinear":

P(z) = DGLAP splitting kernels, with z = energy fraction = $E_a/(E_a+E_b)$

$$|\mathcal{M}_{F+1}(\ldots,a,b,\ldots)|^2 \stackrel{a||b}{\to} g_s^2 \mathcal{C} \frac{P(z)}{2(p_a \cdot p_b)} |\mathcal{M}_F(\ldots,a+b,\ldots)|^2$$

Gluon j → "soft":

Coherence → Parton j really emitted by (i,k) "colour antenna"

$$|\mathcal{M}_{F+1}(\ldots,i,j,k\ldots)|^2 \stackrel{j_g\to 0}{\to} g_s^2 \mathcal{C} \frac{(p_i \cdot p_k)}{(p_i \cdot p_j)(p_j \cdot p_k)} |\mathcal{M}_F(\ldots,i,k,\ldots)|^2$$

+ scaling violation: $g_s^2 \to 4\pi\alpha_s(Q^2)$

See: PS, *Introduction to QCD*, TASI 2012, arXiv:1207.2389

Jets ≈ Fractals

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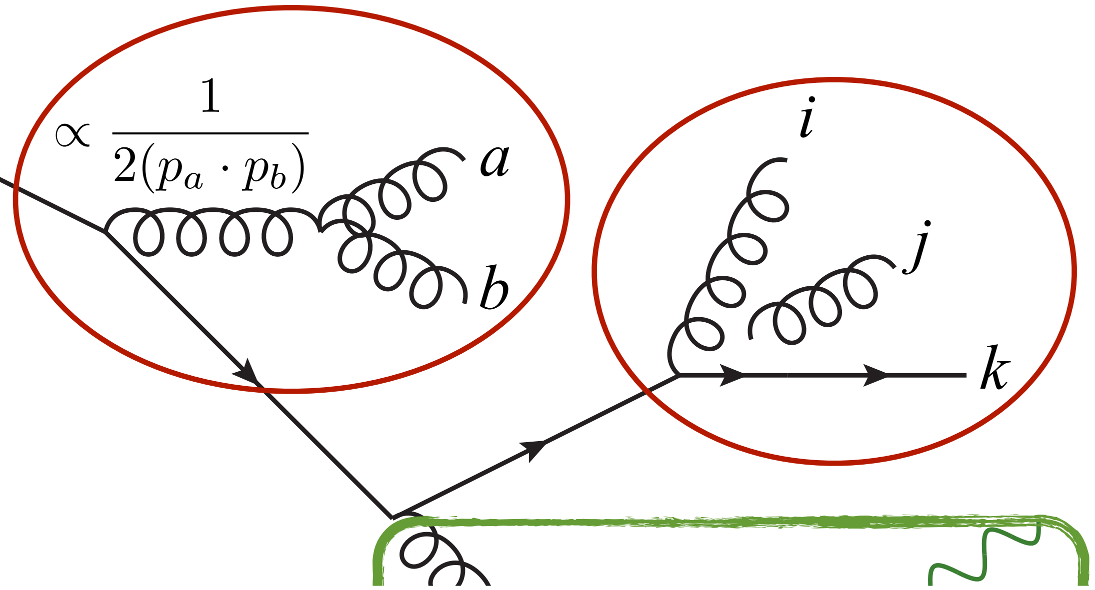

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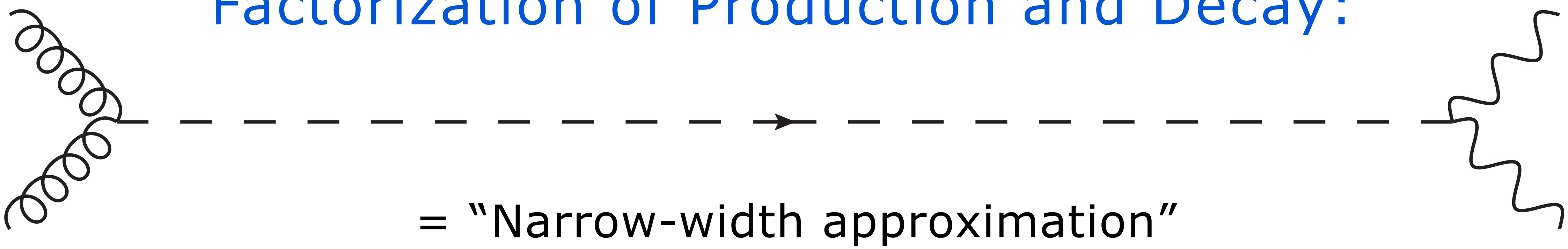
Can apply this many times → nested factorizations

See: PS, *Introduction to QCD*, TASI 2012, arXiv:1207.2389

Factorization: Separation of Scales

Factorization of Production and Decay:

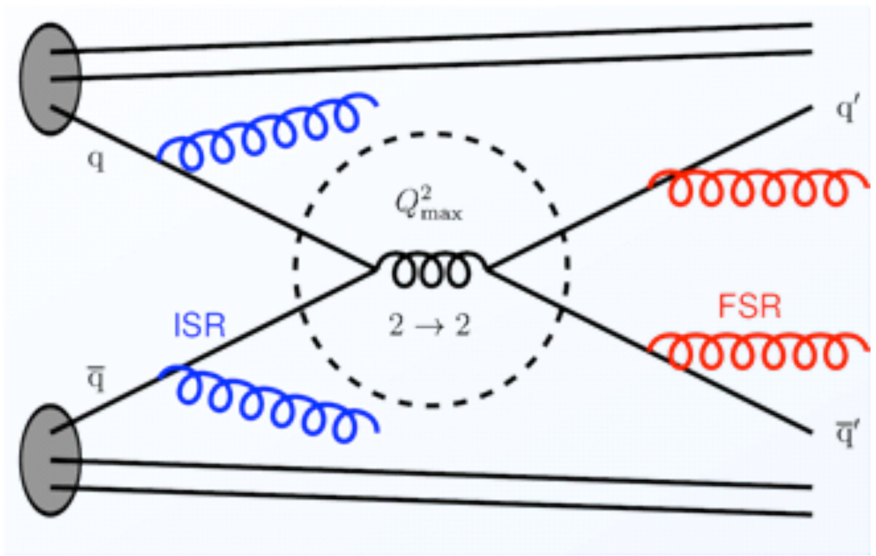


= "Narrow-width approximation"

Valid up to corrections Γ/m → breaks down for large Γ

More subtle when colour/charge flows *through* the diagram

Factorization of Long and Short Distances



Scale of fluctuations inside a hadron

~ Λ_{QCD} ~ 200 MeV

Scale of hard process ≫ Λ_{QCD}

→ proton looks "frozen"

Instantaneous snapshot of long-wavelength structure, independent of nature of hard process

Factorization 2: PDFs

Hadrons are composite, with time-dependent structure:

Partons within clouds of further partons, constantly emitted and absorbed

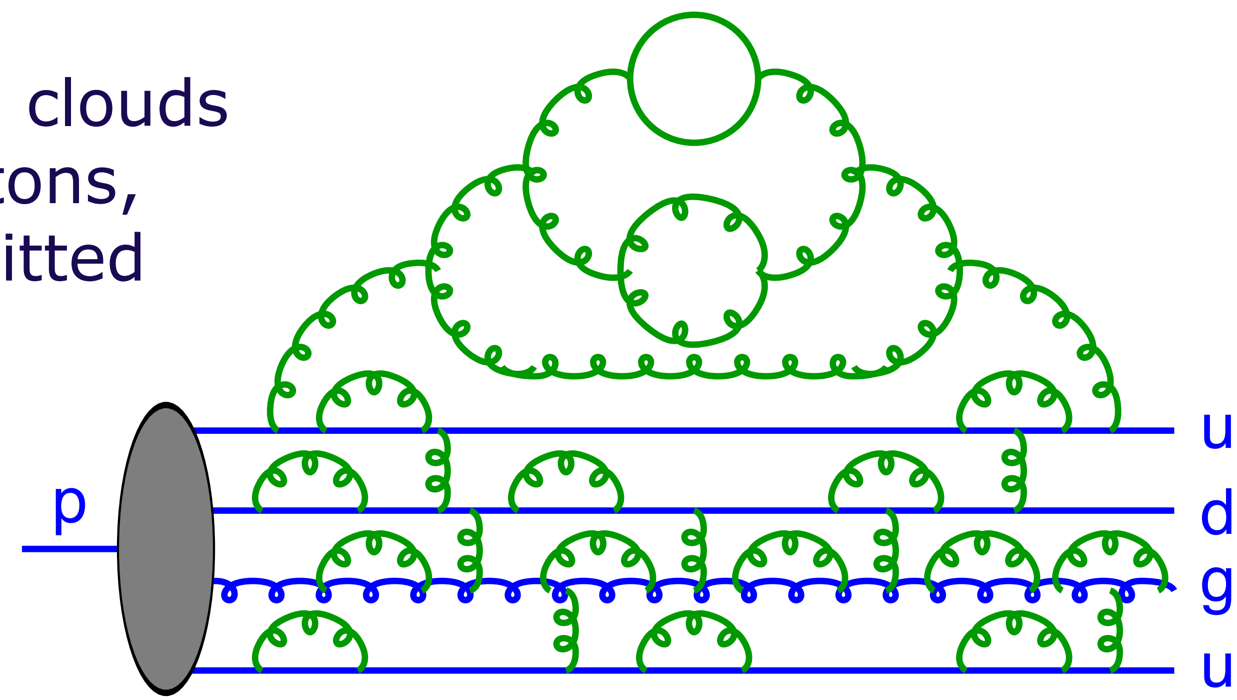


Illustration from T. Sjöstrand

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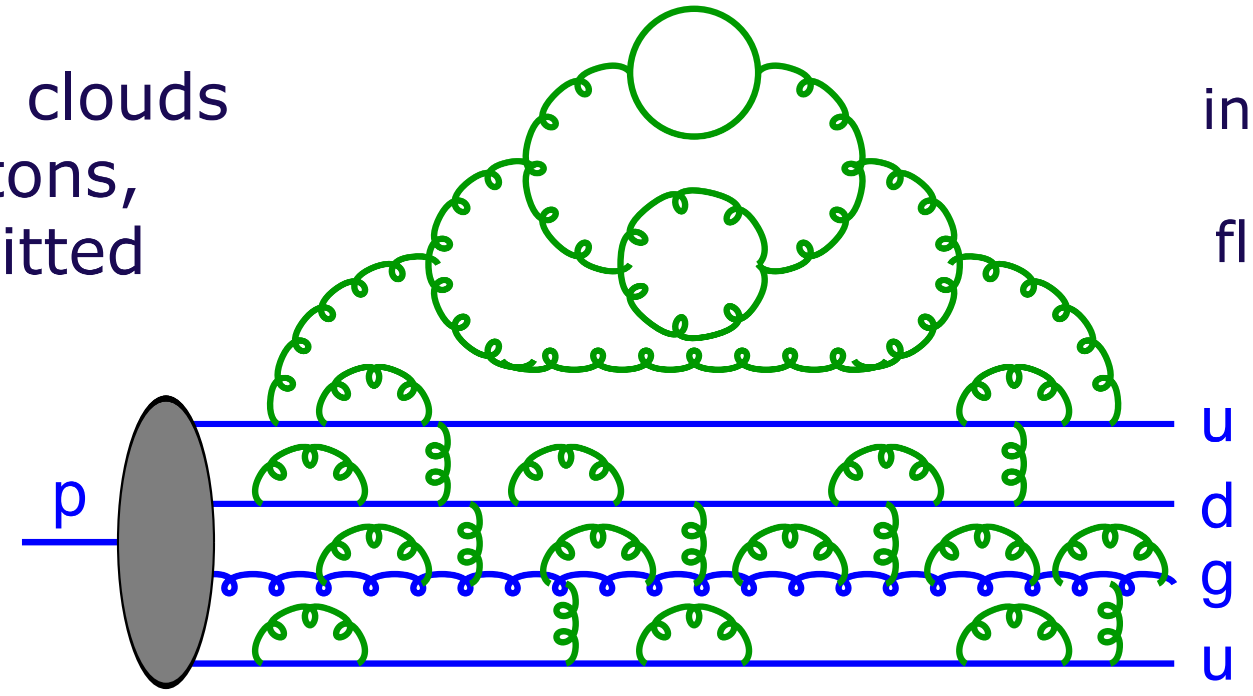

For hadron to remain intact, virtualities $k^2 < M_h^2$
High-virtuality fluctuations suppresed by powers of

$$\frac{\alpha_s M_h^2}{k^2}$$

M_h : mass of hadron
k^2 : virtuality of fluctuation

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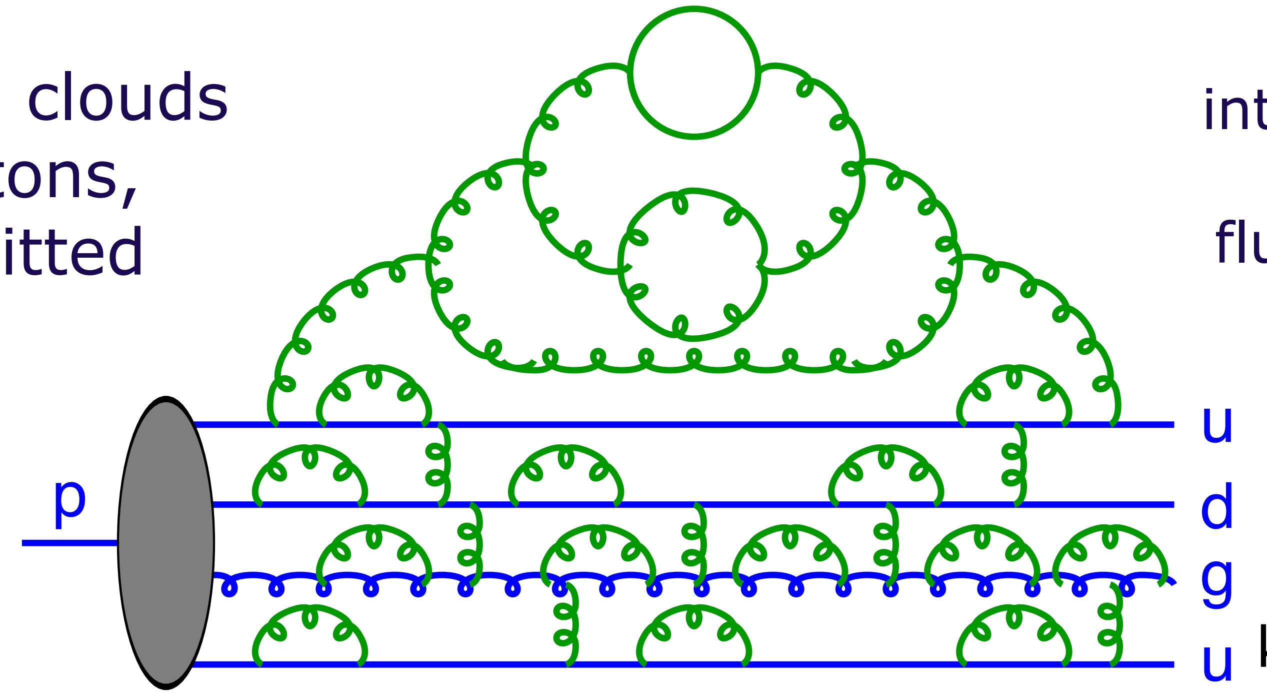

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→ Lifetime of fluctuations ~ $1/M_h$

Hard incoming probe interacts over much shorter time scale ~ 1/Q

On that timescale, partons ~ frozen

Hard scattering knows nothing of the target hadron apart from the fact that it contained the struck parton

Illustration from T. Sjöstrand

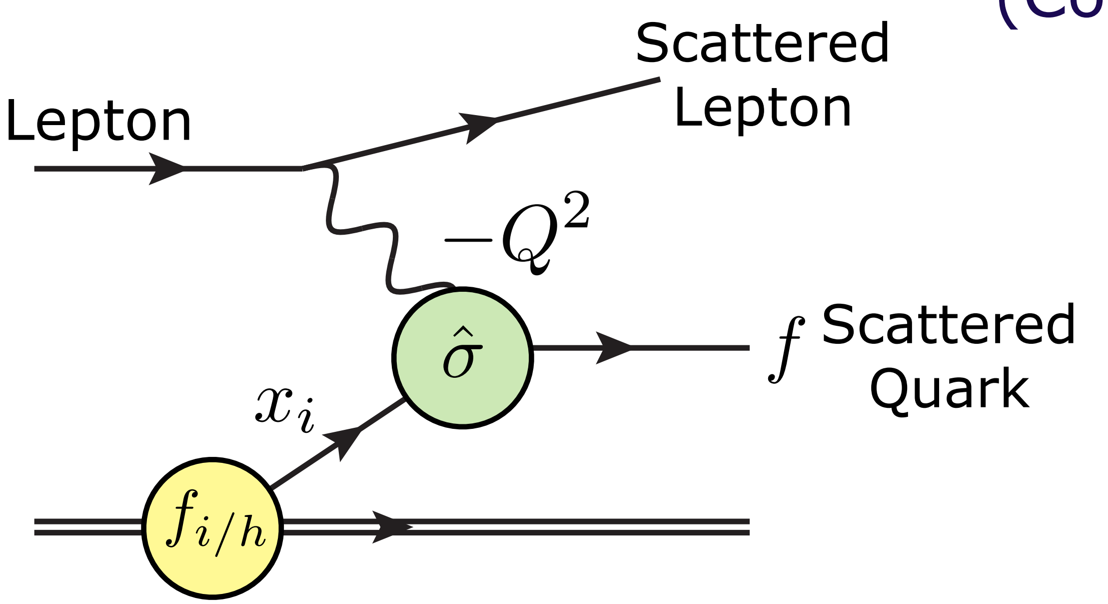
Factorization Theorem

In DIS, there is a formal proof of factorization

(Collins, Soper, 1987)

Deep Inelastic Scattering (DIS)

(By "deep", we mean $Q^2 >> M_h^2$)



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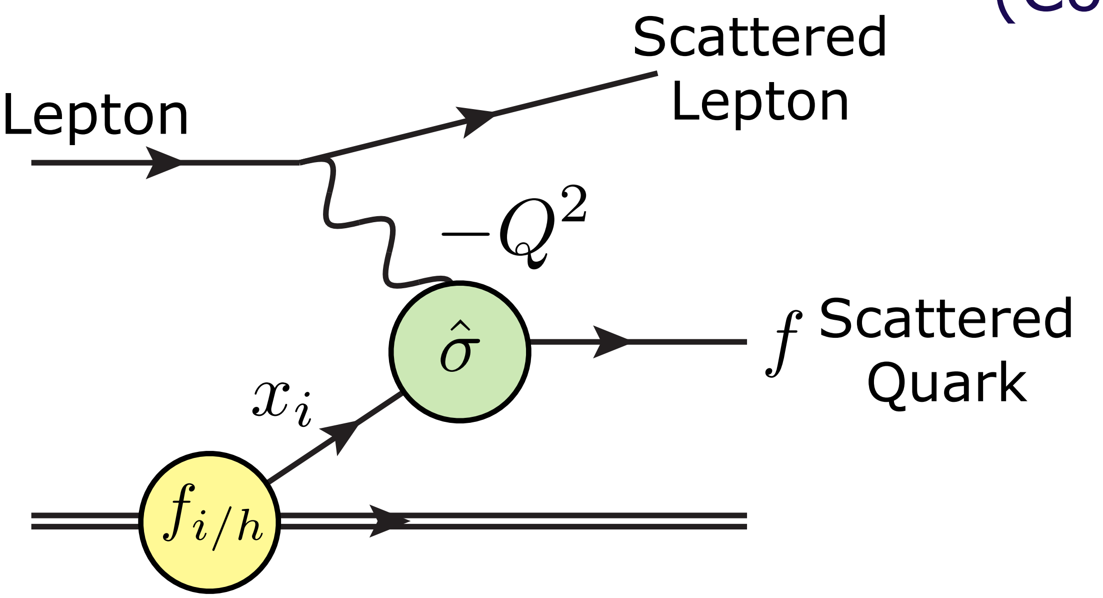

→ We really can write the cross section in factorized form :

$$\sigma^{\ell h} = \sum_i \sum_f \int dx_i \int d\Phi_f \, f_{i/h}(x_i, Q_F^2) \, \frac{d\hat{\sigma}^{\ell i \to f}(x_i, \Phi_f, Q_F^2)}{dx_i \, d\Phi_f}$$

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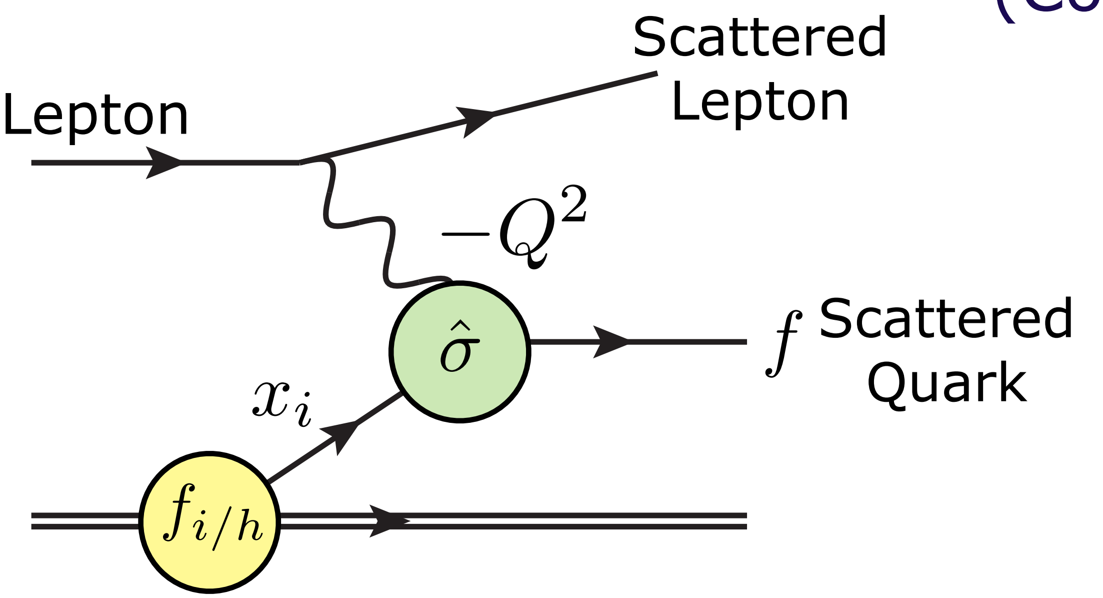

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Sum over Initial (i) and final (f) parton flavors

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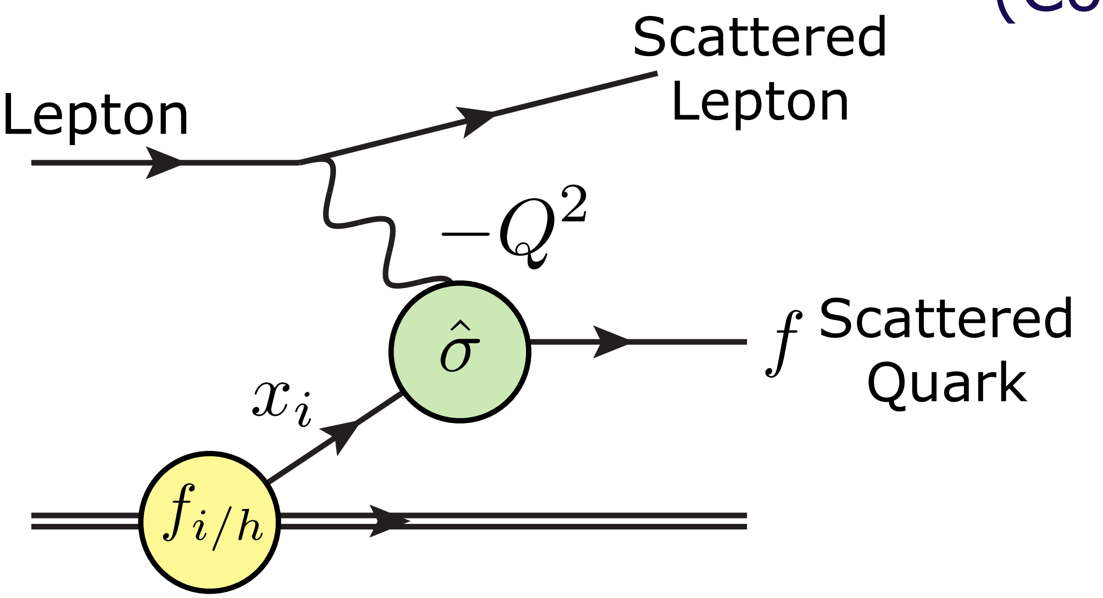

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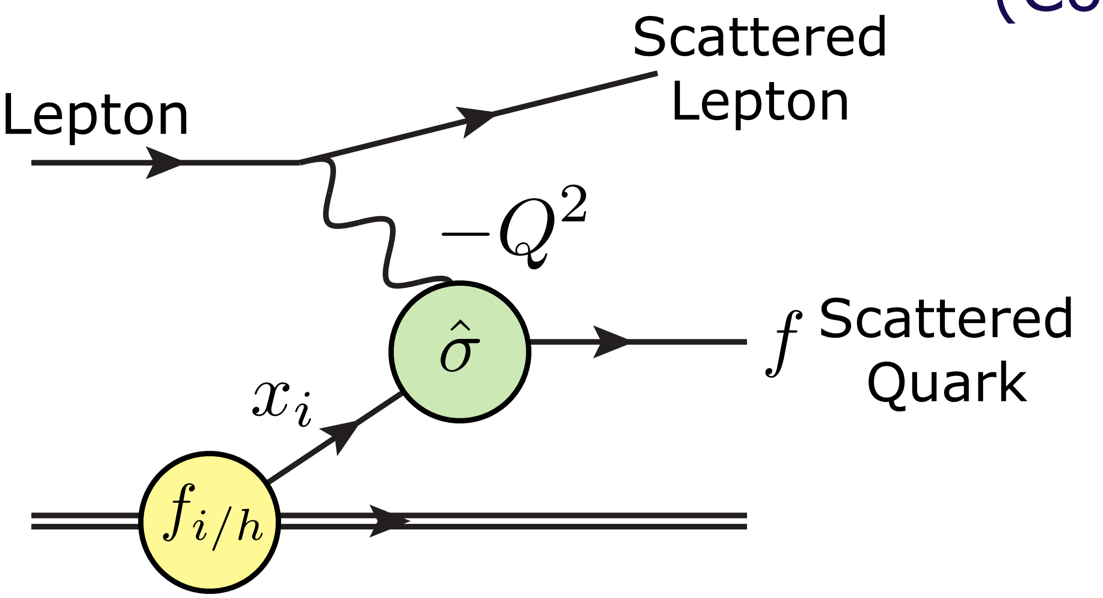

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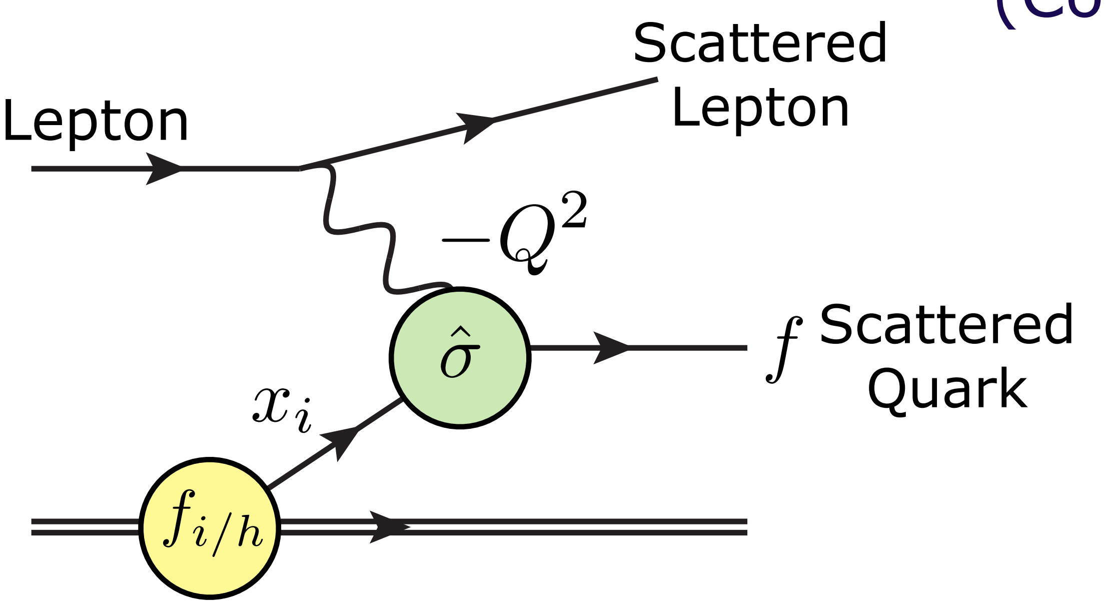

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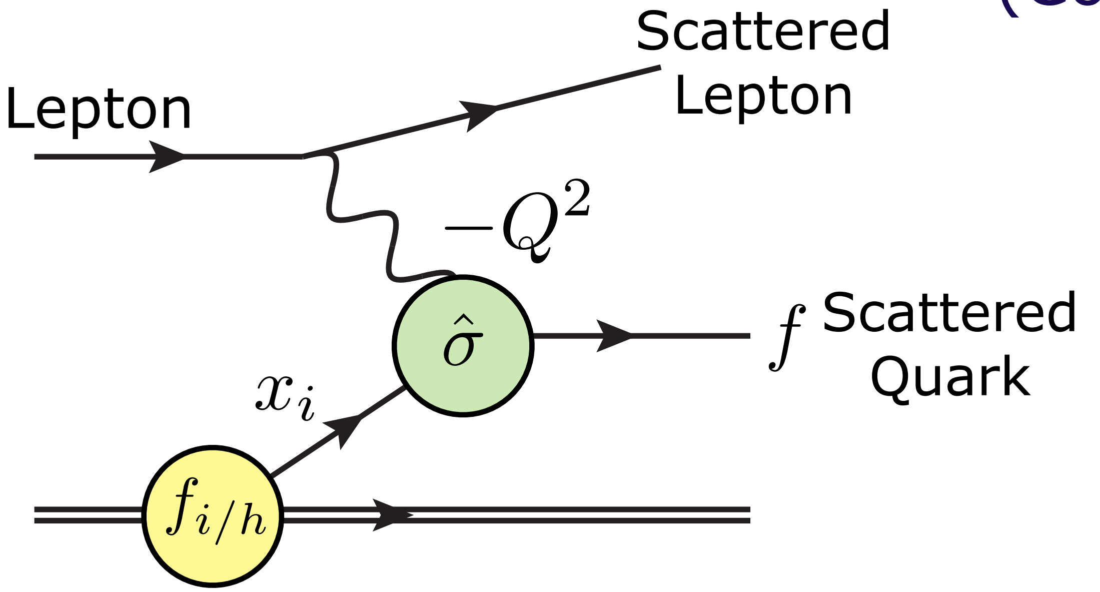

Note: Beyond LO, *f* can be more than one parton

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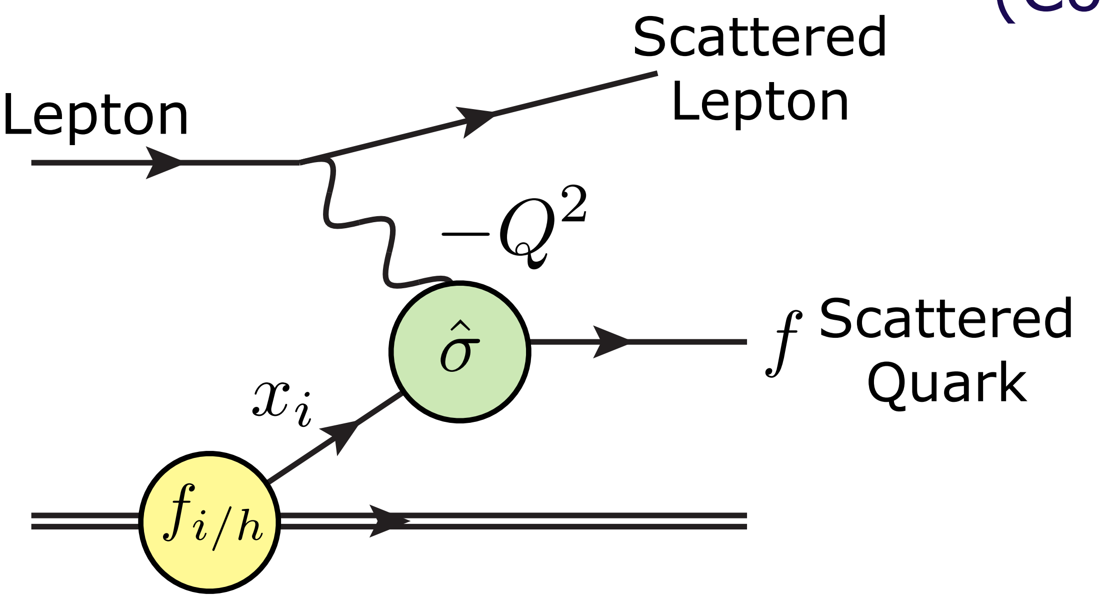

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Surprise Question: What's the color factor for DIS?

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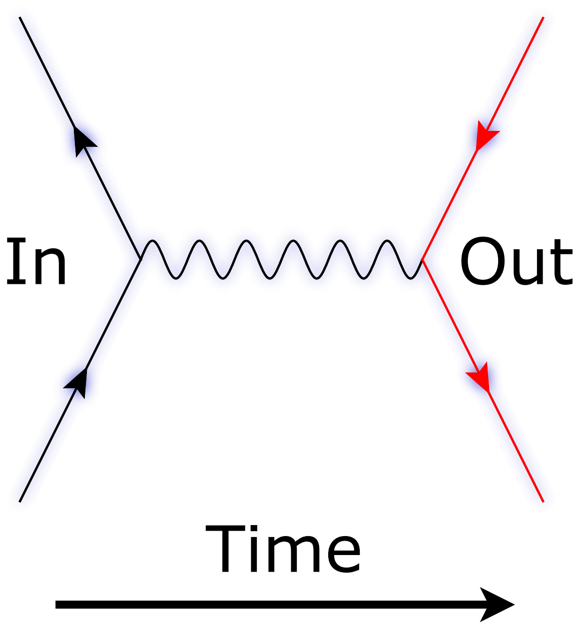
It's just another crossing

$e^+e^- \to \gamma^*/Z \to q\bar{q}$

(Hadronic Z Decay)

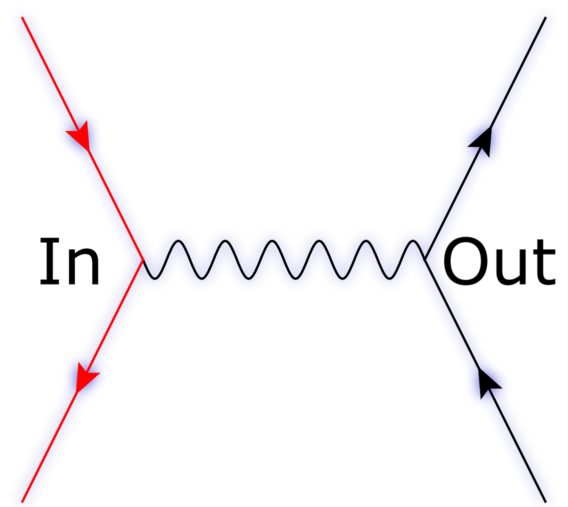


Color Factor:

$$\mathrm{Tr}[\delta_{ij}] = N_C$$

$q\bar{q} \to \gamma^*/Z \to \ell^+\ell^-$

(Drell & Yan, 1970)

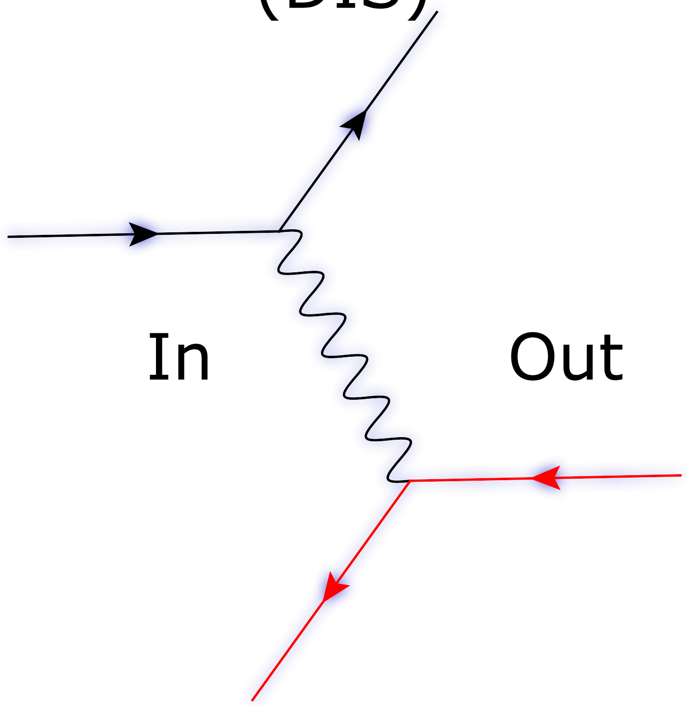


Color Factor:

$$\frac{1}{N_C^2}\mathrm{Tr}[\delta_{ij}] = \frac{1}{N_C}$$

$\ell q \overset{\gamma^*/Z}{\to} \ell q$

(DIS)



Color Factor:

$$\frac{1}{N_C}\mathrm{Tr}[\delta_{ij}] = 1$$

Factorization

Why is Fixed Order QCD not enough?
: It requires all resolved scales >> Λ_{QCD} **AND** no large hierarchies

Trivially untrue for QCD

- We're colliding, and observing, hadrons → small scales
- We want to consider high-scale processes → large scale differences

→ A Priori, no perturbatively calculable observables in hadron-hadron collisions

Factorization

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PDFs: needed to compute inclusive cross sections

FFs: needed to compute (semi-)exclusive cross sections

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FFs: needed to compute (semi-)exclusive cross sections

Resummed pQCD: All resolved scales >> Λ_{QCD} **AND** X Infrared Safe

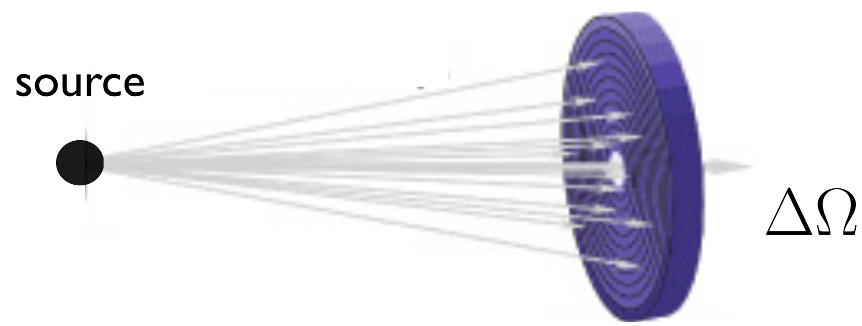
*) pQCD = perturbative QCD

P. Skands
Event Generator Physics
Lecture 2

Terascale Monte Carlo School
DESY Hamburg
March 2014

Introduction to Monte Carlo

Recall: Scattering Experiments



LHC detector
Cosmic-Ray detector
Neutrino detector
X-ray telescope
…

→ Integrate differential cross sections over specific phase-space regions

Predicted number of counts
= integral over solid angle

$$N_{\mathrm{count}}(\Delta\Omega) \propto \int_{\Delta\Omega} \mathrm{d}\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$$

Complicated integrands?
→ Numerical approaches
High-dimensional phase spaces?
→ Monte Carlo

In particle physics:
Integrate over all quantum histories
(+ interferences)

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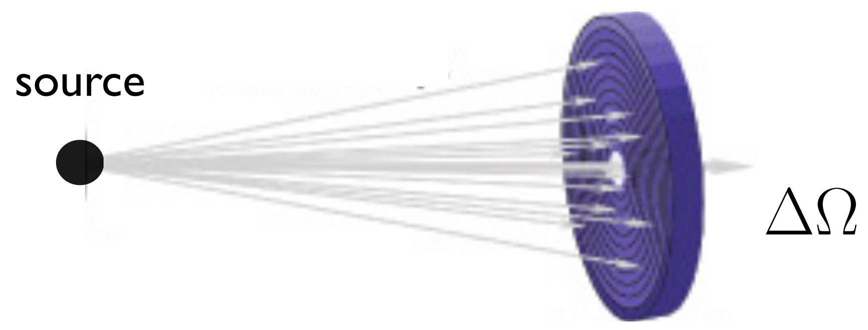

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Why Numerical?

Part of Z → 4 jets …

5.3 Four-parton tree-level antenna functions

The tree-level four-parton quark-antiquark antenna contains three final states: quark-gluon-gluon-antiquark at leading and subleading colour, A_4^0 and $\tilde{A}_4^0$ and quark-antiquark-quark-antiquark for non-identical quark flavours B_4^0 as well as the identical-flavour-only contribution C_4^0. The quark-antiquark-quark-antiquark final state with identical quark flavours is thus described by the sum of antennae for non-identical flavour and identical-flavours-only. The antennae for the $qgg\bar{q}$ final state are:

$$A_4^0(1_q,3_g,4_g,2_{\bar{q}}) = a_4^0(1,3,4,2) + a_4^0(2,4,3,1)\,, \tag{5.27}$$

$$\tilde{A}_4^0(1_q,3_g,4_g,2_{\bar{q}}) = \tilde{a}_4^0(1,3,4,2) + \tilde{a}_4^0(2,4,3,1) + \tilde{a}_4^0(1,4,3,2) + \tilde{a}_4^0(2,3,4,1)\,, \tag{5.28}$$

$$\begin{aligned}
a_4^0(1,3,4,2) = \frac{1}{s_{1234}}\Bigg\{ & \frac{1}{2s_{13}s_{24}s_{34}}\left[2s_{12}s_{14} + 2s_{12}s_{23} + 2s_{12}^2 + s_{14}^2 + s_{23}^2\right] \\
& + \frac{1}{2s_{13}s_{24}s_{134}s_{234}}\left[3s_{12}s_{34}^2 - 4s_{12}^2s_{34} + 2s_{12}^3 - s_{34}^3\right] \\
& + \frac{1}{s_{13}s_{24}s_{134}}\left[3s_{12}s_{23} - 3s_{12}s_{34} + 4s_{12}^2 - s_{23}s_{34} + s_{23}^2 + s_{34}^2\right] \\
& + \frac{3}{2s_{13}s_{24}}\left[2s_{12} + s_{14} + s_{23}\right] + \frac{1}{s_{13}s_{34}}\left[4s_{12} + 3s_{23} + 2s_{24}\right] \\
& + \frac{1}{s_{13}s_{134}^2}\left[s_{12}s_{34} + s_{23}s_{34} + s_{24}s_{34}\right] \\
& + \frac{1}{s_{13}s_{134}s_{234}}\left[3s_{12}s_{24} + 6s_{12}s_{34} - 4s_{12}^2 - 3s_{24}s_{34} - s_{24}^2 - 3s_{34}^2\right] \\
& + \frac{1}{s_{13}s_{134}}\left[-6s_{12} - 3s_{23} - s_{24} + 2s_{34}\right] \\
& + \frac{1}{s_{24}s_{34}s_{134}}\left[2s_{12}s_{14} + 2s_{12}s_{23} + 2s_{12}^2 + 2s_{14}s_{23} + s_{14}^2 + s_{23}^2\right] \\
& + \frac{1}{s_{24}s_{134}}\left[-4s_{12} - s_{14} - s_{23} + s_{34}\right] + \frac{1}{s_{34}^2}\left[s_{12} + 2s_{13} - 2s_{14} - s_{34}\right] \\
& + \frac{1}{s_{34}^2s_{134}^2}\left[2s_{12}s_{14}^2 + 2s_{14}^2s_{23} + 2s_{14}^2s_{24}\right] - \frac{2s_{12}s_{14}s_{24}}{s_{34}^2s_{134}s_{234}} \\
& + \frac{1}{s_{34}^2s_{134}}\left[-2s_{12}s_{14} - 4s_{14}s_{24} + 2s_{14}^2\right] \\
& + \frac{1}{s_{34}s_{134}s_{234}}\left[-2s_{12}s_{14} - 4s_{12}^2 + 2s_{14}s_{24} - s_{14}^2 - s_{24}^2\right] \\
& + \frac{1}{s_{34}s_{134}}\left[-8s_{12} - 2s_{23} - 2s_{24}\right] + \frac{1}{s_{134}^2}\left[s_{12} + s_{23} + s_{24}\right] \\
& + \frac{3}{2s_{134}s_{234}}\left[2s_{12} + s_{14} - s_{24} - s_{34}\right] + \frac{1}{2s_{134}} + \mathcal{O}(\epsilon)\Bigg\}\,,
\end{aligned} \tag{5.29}$$

$$\begin{aligned}
\tilde{a}_4^0(1,3,4,2) = \frac{1}{s_{1234}}\Bigg\{ & \frac{1}{s_{13}s_{24}s_{134}s_{234}}\left[\frac{3}{2}s_{12}s_{34}^2 - 2s_{12}^2s_{34} + s_{12}^3 - \frac{1}{2}s_{34}^3\right] \\
& + \frac{1}{s_{13}s_{24}s_{134}}\left[3s_{12}s_{23} - 3s_{12}s_{34} + 4s_{12}^2 - s_{23}s_{34} + s_{23}^2 + s_{34}^2\right] \\
& + \frac{s_{12}^3}{s_{13}s_{24}(s_{13}+s_{23})(s_{14}+s_{24})} + \frac{1}{s_{13}s_{24}(s_{13}+s_{23})}\left[\frac{1}{2}s_{12}s_{14} + s_{12}^2\right] \\
& + \frac{1}{s_{13}s_{24}(s_{14}+s_{24})}\left[\frac{1}{2}s_{12}s_{23} + s_{12}^2\right] + \frac{1}{s_{13}s_{24}}\left[3s_{12} + \frac{3}{2}s_{14} + \frac{3}{2}s_{23}\right] \\
& + \frac{1}{s_{13}s_{134}^2}\left[s_{12}s_{34} + s_{23}s_{34} + s_{24}s_{34}\right] + \frac{2s_{12}^3}{s_{13}s_{134}s_{234}(s_{13}+s_{23})} \\
& + \frac{1}{s_{13}s_{134}s_{234}}\left[3s_{12}s_{34} - s_{24}s_{34} - 2s_{34}^2\right] \\
& + \frac{1}{s_{13}s_{134}(s_{13}+s_{23})}\left[s_{12}s_{24} + s_{12}s_{34} + 2s_{12}^2\right] \\
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& + \frac{1}{s_{13}(s_{13}+s_{23})(s_{13}+s_{14})}\left[s_{12}s_{24} + 2s_{12}^2\right] \\
& + \frac{1}{s_{13}(s_{14}+s_{24})(s_{13}+s_{14})}\left[s_{12}s_{23} + 2s_{12}^2\right] \\
& + \frac{2s_{12}}{s_{13}(s_{13}+s_{14})} - \frac{2}{s_{13}} + \frac{1}{s_{134}^2}\left[s_{12} + s_{23} + s_{24}\right] \\
& + \frac{1}{s_{134}s_{234}}\left[s_{12} - s_{34}\right] + \frac{1}{s_{134}} + \mathcal{O}(\epsilon)\Bigg\}\,.
\end{aligned} \tag{5.30}$$

+ Additional Subleading Terms …

Why Numerical?

Part of Z → 4 jets …

5.3 Four-parton tree-level antenna functions

The tree-level four-parton quark-antiquark antenna contains three final states: quark-gluon-gluon-antiquark at leading and subleading colour, A_4^0 and $\tilde{A}_4^0$ and quark-antiquark-quark-antiquark for non-identical quark flavours B_4^0 as well as the identical-flavour-only contribution C_4^0. The quark-antiquark-quark-antiquark final state with identical quark flavours is thus described by the sum of antennae for non-identical flavour and identical-flavour-only. The antennae for the $qgg\bar{q}$ final state are:

$$A_4^0(1_q,3_g,4_g,2_{\bar{q}}) = a_4^0(1,3,4,2) + a_4^0(2,4,3,1)\,, \tag{5.27}$$

$$\tilde{A}_4^0(1_q,3_g,4_g,2_{\bar{q}}) = \tilde{a}_4^0(1,3,4,2) + \tilde{a}_4^0(2,4,3,1) + \tilde{a}_4^0(1,4,3,2) + \tilde{a}_4^0(2,3,4,1)\,, \tag{5.28}$$

$$\begin{aligned}
a_4^0(1,3,4,2) = \frac{1}{s_{1234}}\Bigg\{ & \frac{1}{2s_{13}s_{24}s_{34}}\left[2s_{12}s_{14}+2s_{12}s_{23}+2s_{12}^2+s_{14}^2+s_{23}^2\right] \\
&+\frac{1}{2s_{13}s_{24}s_{134}s_{234}}\left[3s_{12}s_{34}^2-4s_{12}^2s_{34}+2s_{12}^3-s_{34}^3\right] \\
&+\frac{1}{s_{13}s_{24}s_{134}}\left[3s_{12}s_{23}-3s_{12}s_{34}+4s_{12}^2-s_{23}s_{34}+s_{23}^2+s_{34}^2\right] \\
&+\frac{3}{2s_{13}s_{24}}\left[2s_{12}+s_{14}+s_{23}\right]+\frac{1}{s_{13}s_{34}}\left[4s_{12}+3s_{23}+2s_{24}\right] \\
&+\frac{1}{s_{13}s_{134}^2}\left[s_{12}s_{34}+s_{23}s_{34}+s_{24}s_{34}\right] \\
&+\frac{1}{s_{13}s_{134}s_{234}}\left[3s_{12}s_{24}+6s_{12}s_{34}-4s_{12}^2-3s_{24}s_{34}-s_{24}^2-3s_{34}^2\right] \\
&+\frac{1}{s_{13}s_{134}}\left[-6s_{12}-3s_{23}-s_{24}+2s_{34}\right] \\
&+\frac{1}{s_{24}s_{34}s_{134}}\left[2s_{12}s_{14}+2s_{12}s_{23}+2s_{12}^2+2s_{14}s_{23}+s_{14}^2+s_{23}^2\right] \\
&+\frac{1}{s_{24}s_{134}}\left[-4s_{12}-s_{14}-s_{23}+s_{34}\right]+\frac{1}{s_{34}^2}\left[s_{12}+2s_{13}-2s_{14}-s_{34}\right] \\
&+\frac{1}{s_{34}^2s_{134}^2}\left[2s_{12}s_{14}^2+2s_{14}^2s_{23}+2s_{14}^2s_{24}\right]-\frac{2s_{12}s_{14}s_{24}}{s_{34}^2s_{134}s_{234}} \\
&+\frac{1}{s_{34}^2s_{134}}\left[-2s_{12}s_{14}-4s_{14}s_{24}+2s_{14}^2\right] \\
&+\frac{1}{s_{34}s_{134}s_{234}}\left[-2s_{12}s_{14}-4s_{12}^2+2s_{14}s_{24}-s_{14}^2-s_{24}^2\right] \\
&+\frac{1}{s_{34}s_{134}}\left[-8s_{12}-2s_{23}-2s_{24}\right]+\frac{1}{s_{134}^2}\left[s_{12}+s_{23}+s_{24}\right] \\
&+\frac{3}{2s_{134}s_{234}}\left[2s_{12}+s_{14}-s_{24}-s_{34}\right]+\frac{1}{2s_{134}}+\mathcal{O}(\epsilon)\Bigg\}\,,
\end{aligned} \tag{5.29}$$

$$\begin{aligned}
\tilde{a}_4^0(1,3,4,2) = \frac{1}{s_{1234}}\Bigg\{ & \frac{1}{s_{13}s_{24}s_{134}s_{234}}\left[\frac{3}{2}s_{12}s_{34}^2-2s_{12}^2s_{34}+s_{12}^3-\frac{1}{2}s_{34}^3\right] \\
&+\frac{1}{s_{13}s_{24}s_{134}}\left[3s_{12}s_{23}-3s_{12}s_{34}+4s_{12}^2-s_{23}s_{34}+s_{23}^2+s_{34}^2\right] \\
&\quad s_{12}^3 \quad\quad 1 \quad\quad \left[1 \right.
\end{aligned}$$

This is one of the simplest processes … computed at lowest order in the theory.

$$\begin{aligned}
& \quad s_{13}s_{134}s_{234} \\
&+\frac{1}{s_{13}s_{134}(s_{13}+s_{23})}\left[s_{12}s_{24}+s_{12}s_{34}+2s_{12}^2\right] \\
&+\frac{1}{s_{13}s_{134}}\left[-s_{23}-s_{24}+2s_{34}\right]+\frac{1}{s_{13}s_{234}(s_{13}+s_{23})}\left[s_{12}s_{14}+s_{12}s_{34}+2s_{12}^2\right] \\
&+\frac{1}{s_{13}s_{234}}\left[-2s_{12}-2s_{14}+s_{24}+2s_{34}\right] \\
&+\frac{2s_{12}^3}{s_{13}(s_{13}+s_{23})(s_{14}+s_{24})(s_{13}+s_{14})} \\
&+\frac{1}{s_{13}(s_{13}+s_{23})(s_{13}+s_{14})}\left[s_{12}s_{24}+2s_{12}^2\right] \\
&+\frac{1}{s_{13}(s_{14}+s_{24})(s_{13}+s_{14})}\left[s_{12}s_{23}+2s_{12}^2\right] \\
&+\frac{2s_{12}}{s_{13}(s_{13}+s_{14})}-\frac{2}{s_{13}}+\frac{1}{s_{134}^2}\left[s_{12}+s_{23}+s_{24}\right] \\
&+\frac{1}{s_{134}s_{234}}\left[s_{12}-s_{34}\right]+\frac{1}{s_{134}}+\mathcal{O}(\epsilon)\Bigg\}\,.
\end{aligned} \tag{5.30}$$

+ Additional Subleading Terms …

Why Numerical?

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& + \frac{3}{2s_{13}s_{24}}\left[2s_{12}+s_{14}+s_{23}\right] + \frac{1}{s_{13}s_{34}}\left[4s_{12}+3s_{23}+2s_{24}\right] \\
& + \frac{1}{s_{13}s_{134}^2}\left[s_{12}s_{34}+s_{23}s_{34}+s_{24}s_{34}\right] \\
& + \frac{1}{s_{13}s_{134}s_{234}}\left[3s_{12}s_{24}+6s_{12}s_{34}-4s_{12}^2-3s_{24}s_{34}-s_{24}^2-3s_{34}^2\right] \\
& + \frac{1}{s_{13}s_{134}}\left[-6s_{12}-3s_{23}-s_{24}+2s_{34}\right] \\
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& + \frac{1}{s_{34}^2s_{134}}\left[-2s_{12}s_{14}-4s_{14}s_{24}+2s_{14}^2\right] \\
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\end{aligned}$$

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$$\begin{aligned}
& + \frac{1}{s_{13}s_{134}(s_{13}+s_{23})}\left[s_{12}s_{24}+s_{12}s_{34}+2s_{12}^2\right]
\end{aligned}$$

Now compute and add the quantum corrections …

$$\begin{aligned}
& + \frac{\ldots}{s_{13}(s_{13}+s_{23})(s_{14}+s_{24})(s_{13}+s_{14})} \\
& + \frac{1}{s_{13}(s_{13}+s_{23})(s_{13}+s_{14})}\left[s_{12}s_{24}+2s_{12}^2\right] \\
& + \frac{1}{s_{13}(s_{14}+s_{24})(s_{13}+s_{14})}\left[s_{12}s_{23}+2s_{12}^2\right] \\
& + \frac{2s_{12}}{s_{13}(s_{13}+s_{14})} - \frac{2}{s_{13}} + \frac{1}{s_{134}^2}\left[s_{12}+s_{23}+s_{24}\right] \\
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&+\frac{3}{2s_{13}s_{24}}\left[2s_{12}+s_{14}+s_{23}\right]+\frac{1}{s_{13}s_{34}}\left[4s_{12}+3s_{23}+2s_{24}\right] \\
&+\frac{1}{s_{13}s_{134}^2}\left[s_{12}s_{34}+s_{23}s_{34}+s_{24}s_{34}\right] \\
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&+\frac{1}{s_{24}s_{134}}\left[-4s_{12}-s_{14}-s_{23}+s_{34}\right]+\frac{1}{s_{34}^2}\left[s_{12}+2s_{13}-2s_{14}-s_{34}\right] \\
&+\frac{1}{s_{34}^2s_{134}^2}\left[2s_{12}s_{14}^2+2s_{14}^2s_{23}+2s_{14}^2s_{24}\right]-\frac{2s_{12}s_{14}s_{24}}{s_{34}^2s_{134}s_{234}} \\
&+\frac{1}{s_{34}^2s_{134}}\left[-2s_{12}s_{14}-4s_{14}s_{24}+2s_{14}^2\right] \\
&+\frac{1}{s_{34}s_{134}s_{234}}\left[-2s_{12}s_{14}-4s_{12}^2+2s_{14}s_{24}-s_{14}^2-s_{24}^2\right] \\
&+\frac{1}{s_{34}s_{134}}\left[-8s_{12}-2s_{23}-2s_{24}\right]+\frac{1}{s_{134}^2}\left[s_{12}+s_{23}+s_{24}\right] \\
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&\ldots
\end{aligned}$$

This is one of the simplest processes … computed at lowest order in the theory.

$$+\frac{1}{s_{13}s_{134}(s_{13}+s_{23})}\left[s_{12}s_{24}+s_{12}s_{34}+2s_{12}^2\right]$$

Now compute and add the quantum corrections …

$$+\frac{\ldots}{s_{13}(s_{13}+s_{23})(s_{14}+s_{24})(s_{13}+s_{14})}$$

$$+\frac{1}{s_{13}(s_{13}+s_{23})(s_{13}+s_{14})}\left[s_{12}s_{24}+2s_{12}^2\right]$$

Then maybe worry about simulating the detector too …

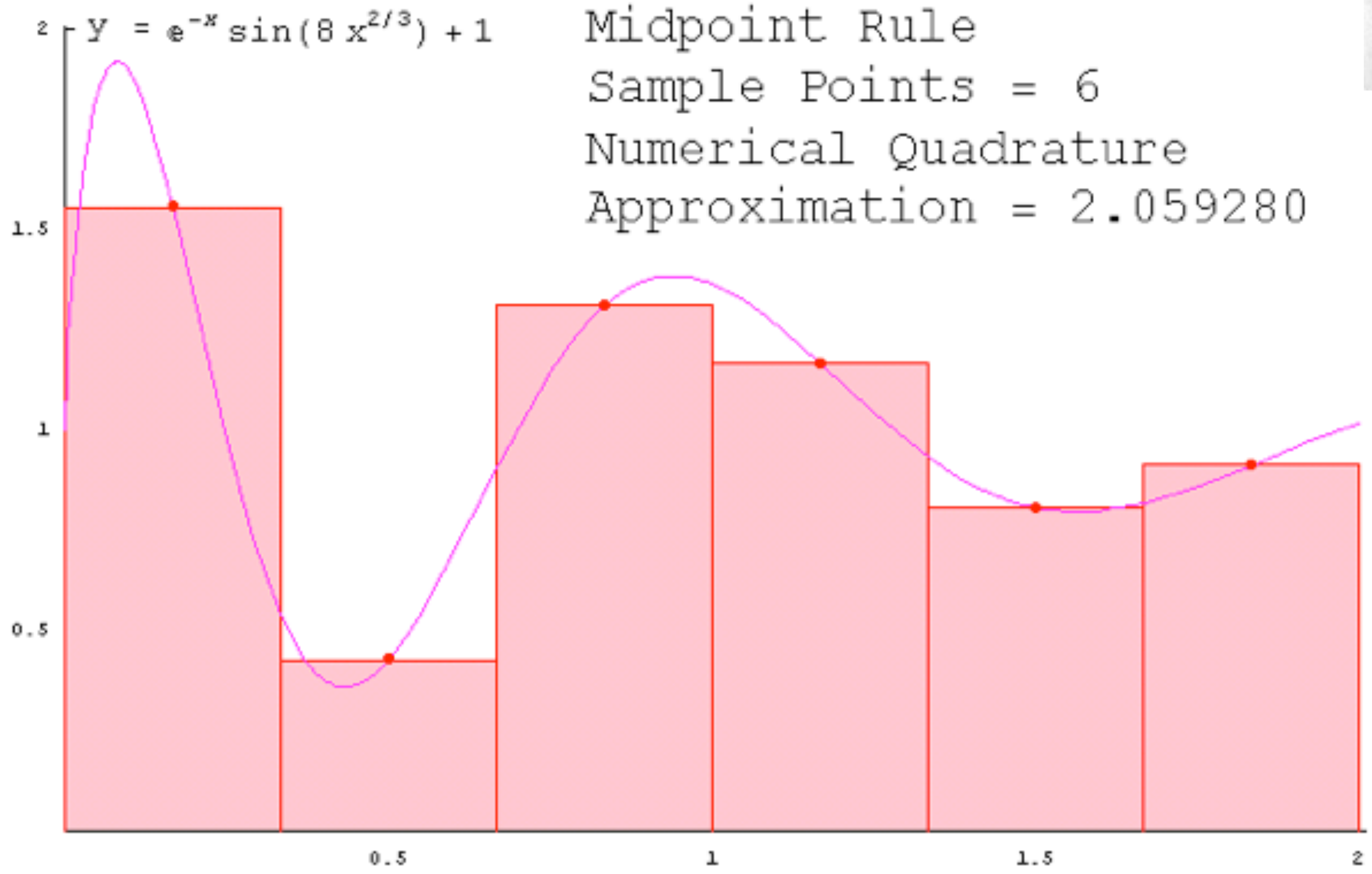
(5.30)

+ Additional Subleading Terms …

Riemann Sums

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)(x_{i+1}-x_i)$$

B. Riemann, (1826-1866)



Numerical Integration in 1D

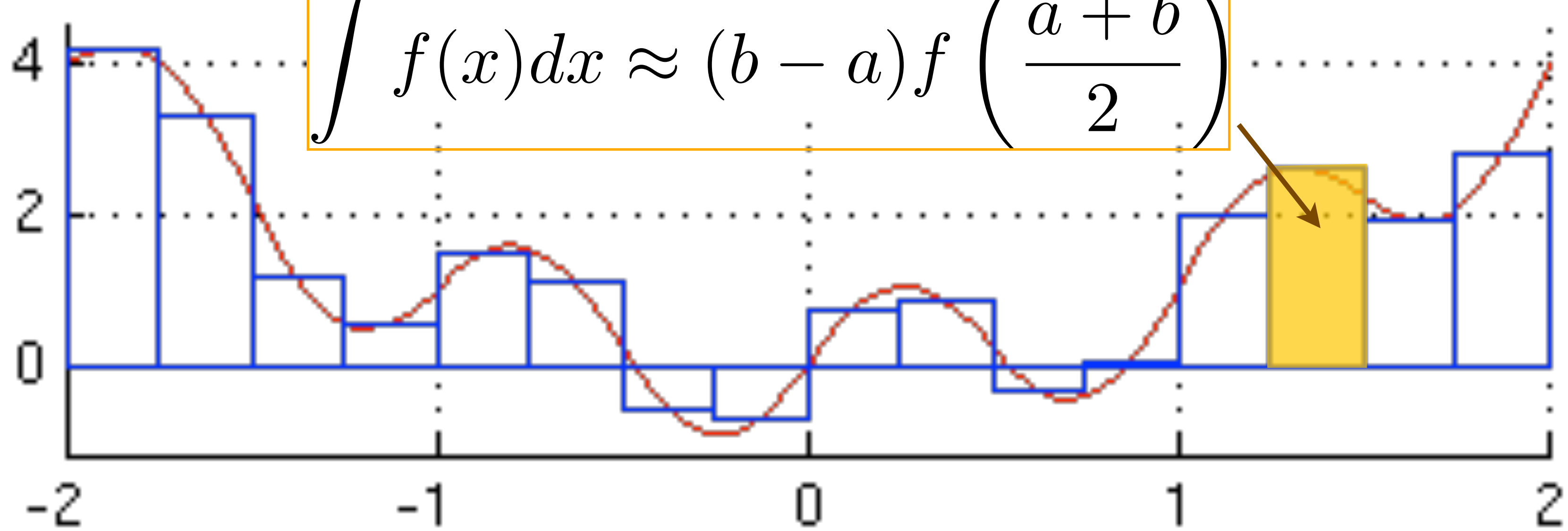
Midpoint (rectangular) Rule:

Fixed-Grid n-point Quadrature Rules

Divide into N "bins" of size Δ
Approximate f(x) ≈ constant in each bin
Sum over all rectangles inside your region

1 function evaluation per bin

$$\int f(x)dx \approx (b-a) f\left(\frac{a+b}{2}\right)$$



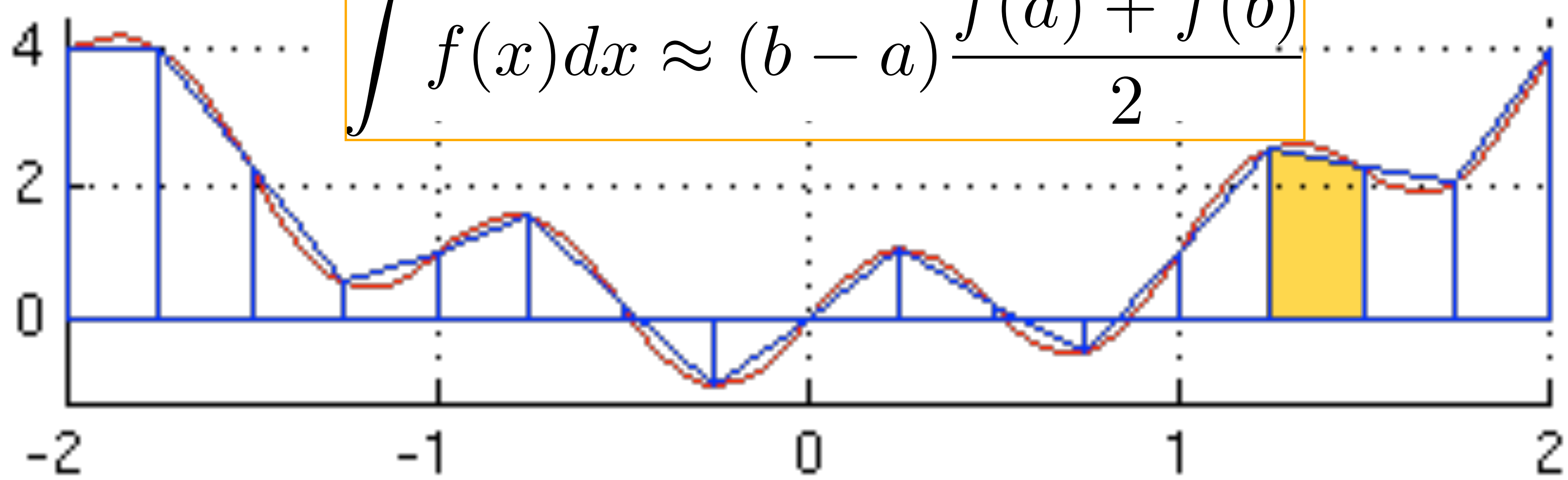
<u>Trapezoidal Rule</u>:

Fixed-Grid n-point Quadrature Rules

Approximate f(x) ≈ linear in each bin
Sum over all trapeziums inside your region

2 function evaluations per bin

$$\int f(x)dx \approx (b-a)\frac{f(a)+f(b)}{2}$$



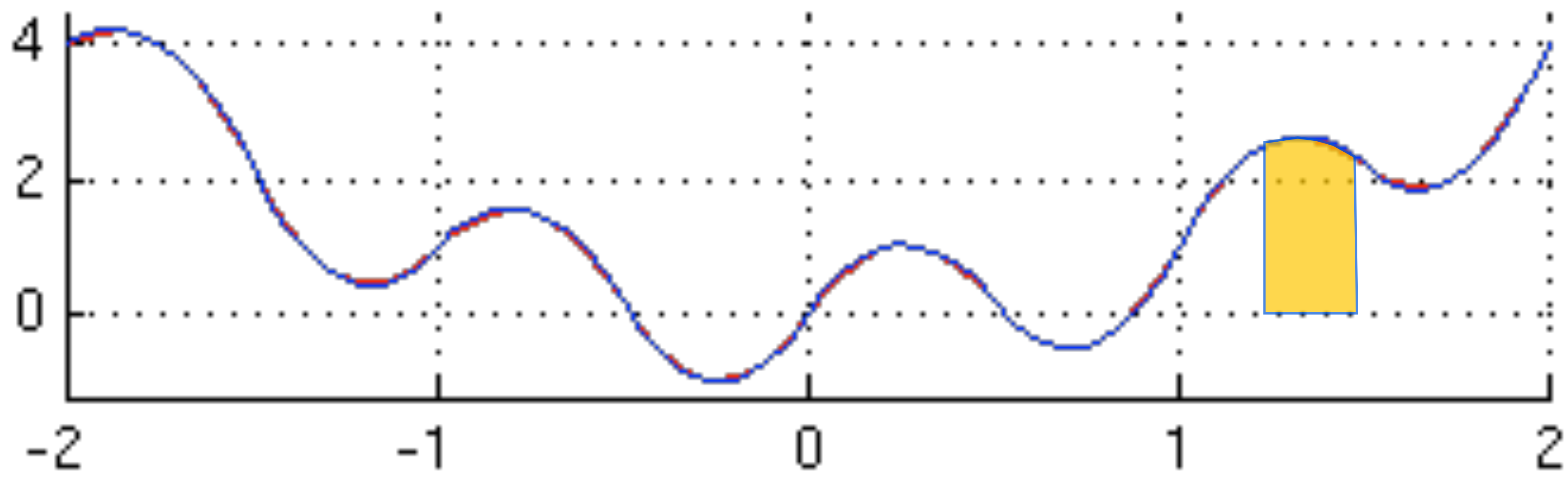
Numerical Integration in 1D

Fixed-Grid n-point Quadrature Rules

Simpson's Rule:

Approximate f(x) ≈ quadratic in each bin
Sum over all "Simpsons" inside your region

3 function evaluations per bin



… and so on for higher n-point rules ...

Convergence Rate

The most important question:

How long do I have to wait?

How many evaluations do I need to calculate for a given precision?

Uncertainty (after n evaluations)	n_{eval} / bin	Approx Conv. Rate (in 1D)
Trapezoidal Rule (2-point)	2	$1/N^2$
Simpson's Rule (3-point)	3	$1/N^4$
… m-point (Gauss quadrature)	m	$1/N^{2m-1}$

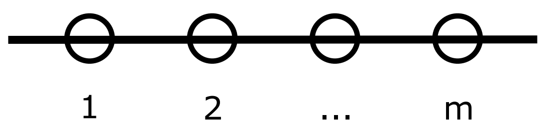
See, e.g., Numerical Recipes

See, e.g., F. James, "Monte Carlo Theory and Practice"

Higher Dimensions

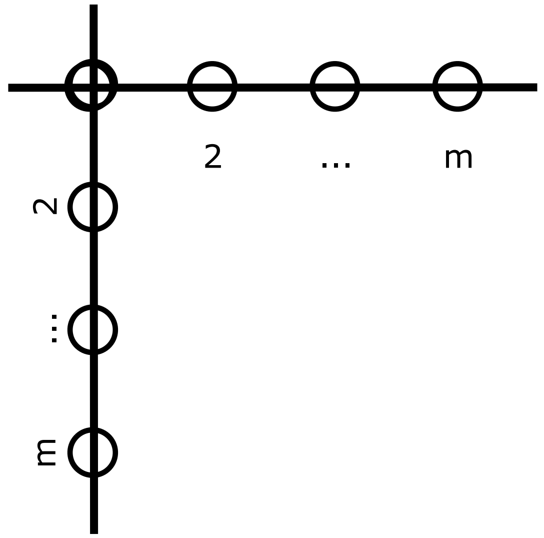
Fixed-Grid (Product) Rules scale exponentially with D

m-point rule in 1 dimension



→ m function evaluations per bin

… in 2 dimensions

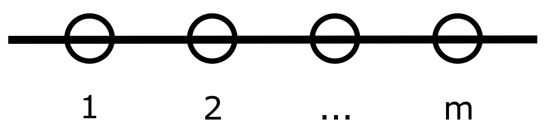


→ m^2 evaluations per bin

Higher Dimensions

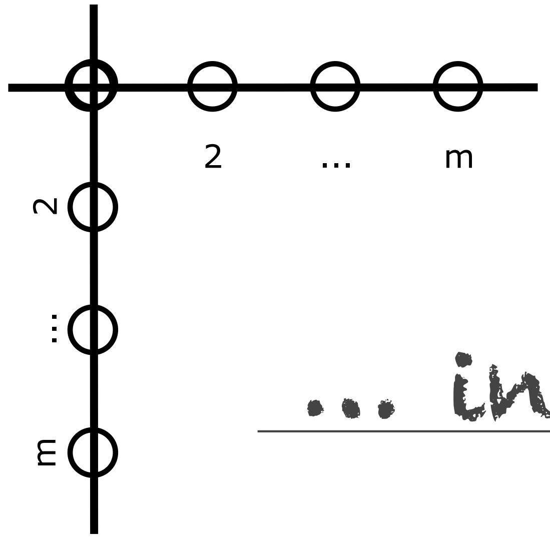
Fixed-Grid (Product) Rules scale exponentially with D

m-point rule in 1 dimension



→ m function evaluations per bin

... in 2 dimensions



→ m^2 evaluations per bin

... in D dimensions → m^D per bin

E.g., to evaluate a 12-point rule in 10 dimensions, need 1000 billion evaluations per bin

Convergence Rate

+ Convergence is slower in higher Dimensions!

→ More points for less precision

Uncertainty (after n evaluations)	n_{eval} / bin	Approx Conv. Rate (in D dim)
Trapezoidal Rule (2-point)	2^D	$1/n^{2/D}$
Simpson's Rule (3-point)	3^D	$1/n^{4/D}$
… m-point (Gauss rule)	m^D	$1/n^{(2m-1)/D}$

See, e.g., Numerical Recipes

See, e.g., F. James, "Monte Carlo Theory and Practice"


→ Monte Carlo

A Monte Carlo technique: is any technique making use of random numbers to solve a problem

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Monte Carlo

Convergence:

Calculus: {A} converges to B
if an n exists for which
$|A_{i>n} - B| < \varepsilon$, for any $\varepsilon > 0$

Monte Carlo: {A} converges to B
if n exists for which
the probability for
$|A_{i>n} - B| < \varepsilon$, for any $\varepsilon > 0$,
is > P, for any P[0<P<1]

A Monte Carlo technique: is any technique making use of random numbers to solve a problem

Convergence:

Calculus: {A} converges to B
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is > P, for any P[0<P<1]

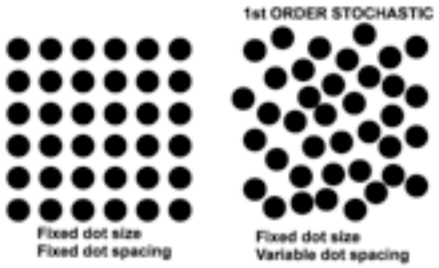
"This risk, that convergence is only given with a certain probability, is inherent in Monte Carlo calculations and is the reason why this technique was named after the world's most famous gambling casino. Indeed, the name is doubly appropriate because the style of gambling in the Monte Carlo casino, not to be confused with the noisy and tasteless gambling houses of Las Vegas and Reno, is serious and sophisticated."

F. James, "Monte Carlo theory and practice", Rept. Prog. Phys. 43 (1980) 1145

Convergence

MC convergence is Stochastic!

$\frac{1}{\sqrt{n}}$ in <u>any</u> dimension



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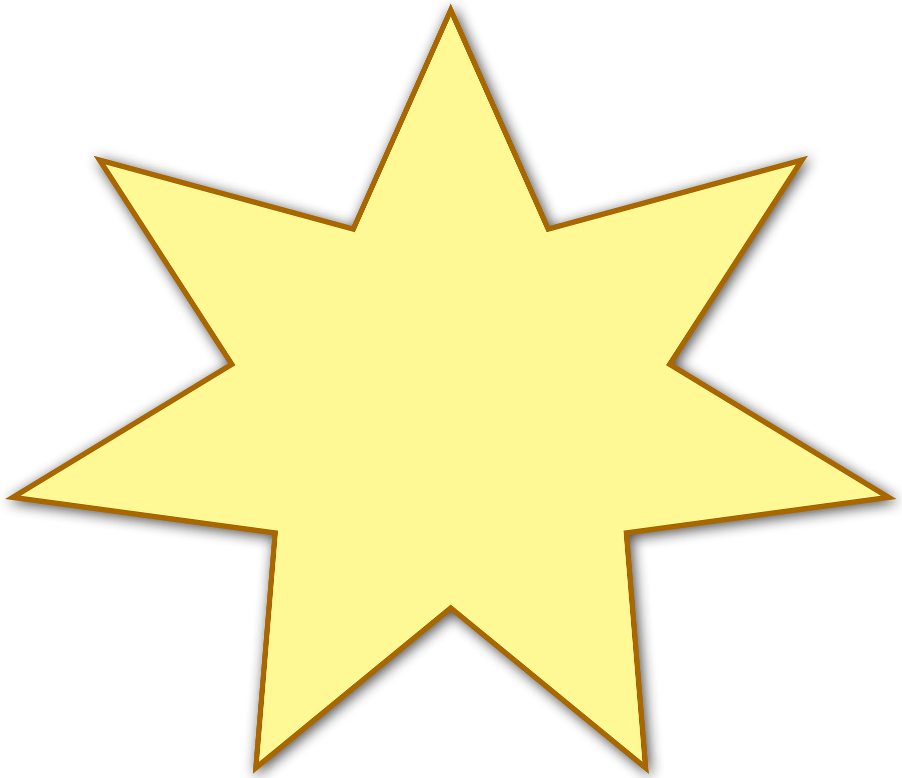
Simpson's Rule (3-point)	3^D	$1/n^4$	$1/n^{4/D}$
… m-point (Gauss rule)	m^D	$1/n^{2m-1}$	$1/n^{(2m-1)/D}$
Monte Carlo	1	$1/n^{1/2}$	$1/n^{1/2}$

+ many ways to optimize: stratification, adaptation, …
+ gives "events" → iterative solutions,
+ interfaces to detector simulation & propagation codes

Random Numbers

(apologies, I did not have much time to adapt these slides)

You want: to know the area of this shape:



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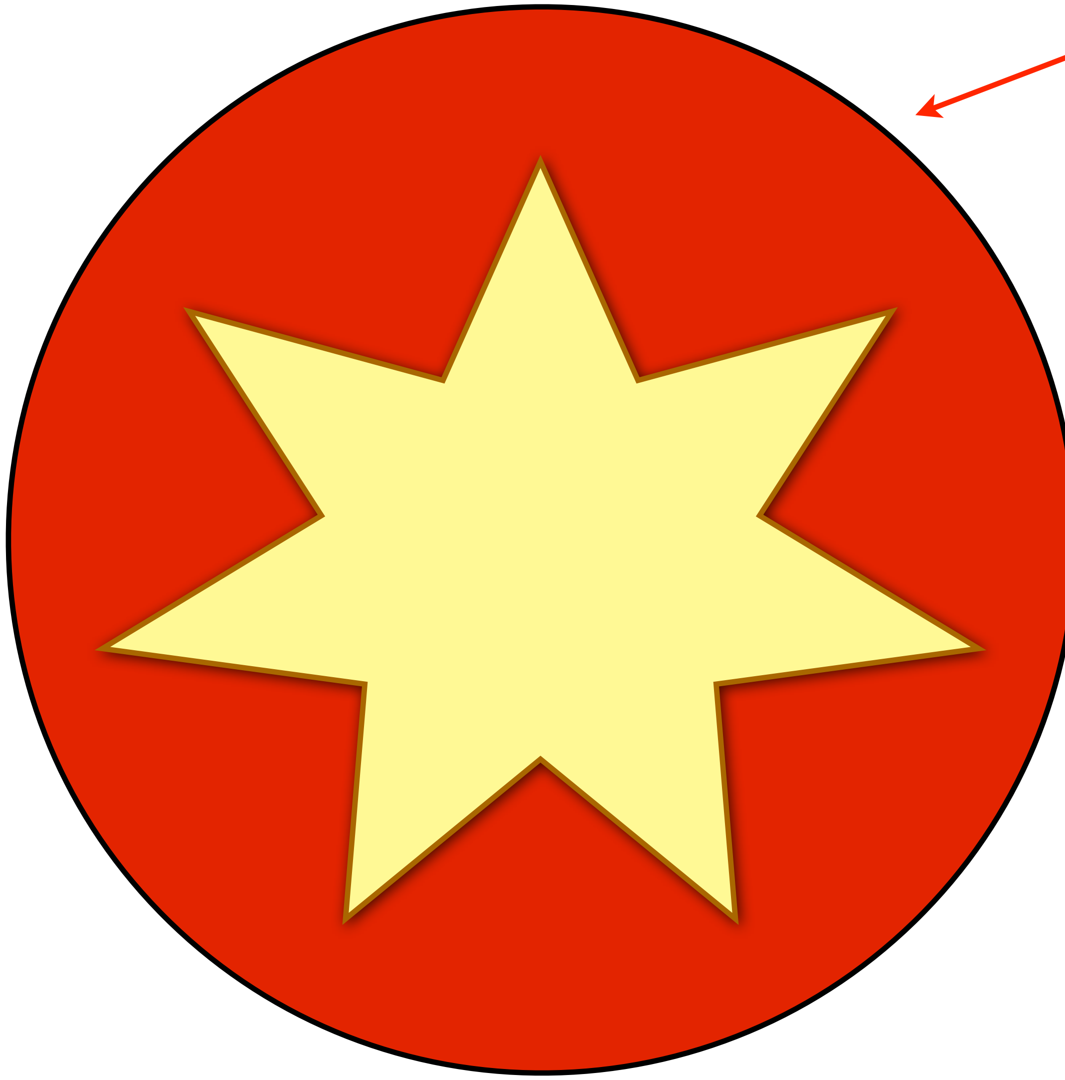

Assume you know the area of this shape: πR^2 (an overestimate)

Random Numbers

(apologies, I did not have much time to adapt these slides)

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Now get a few friends, some balls, and throw random shots inside the circle
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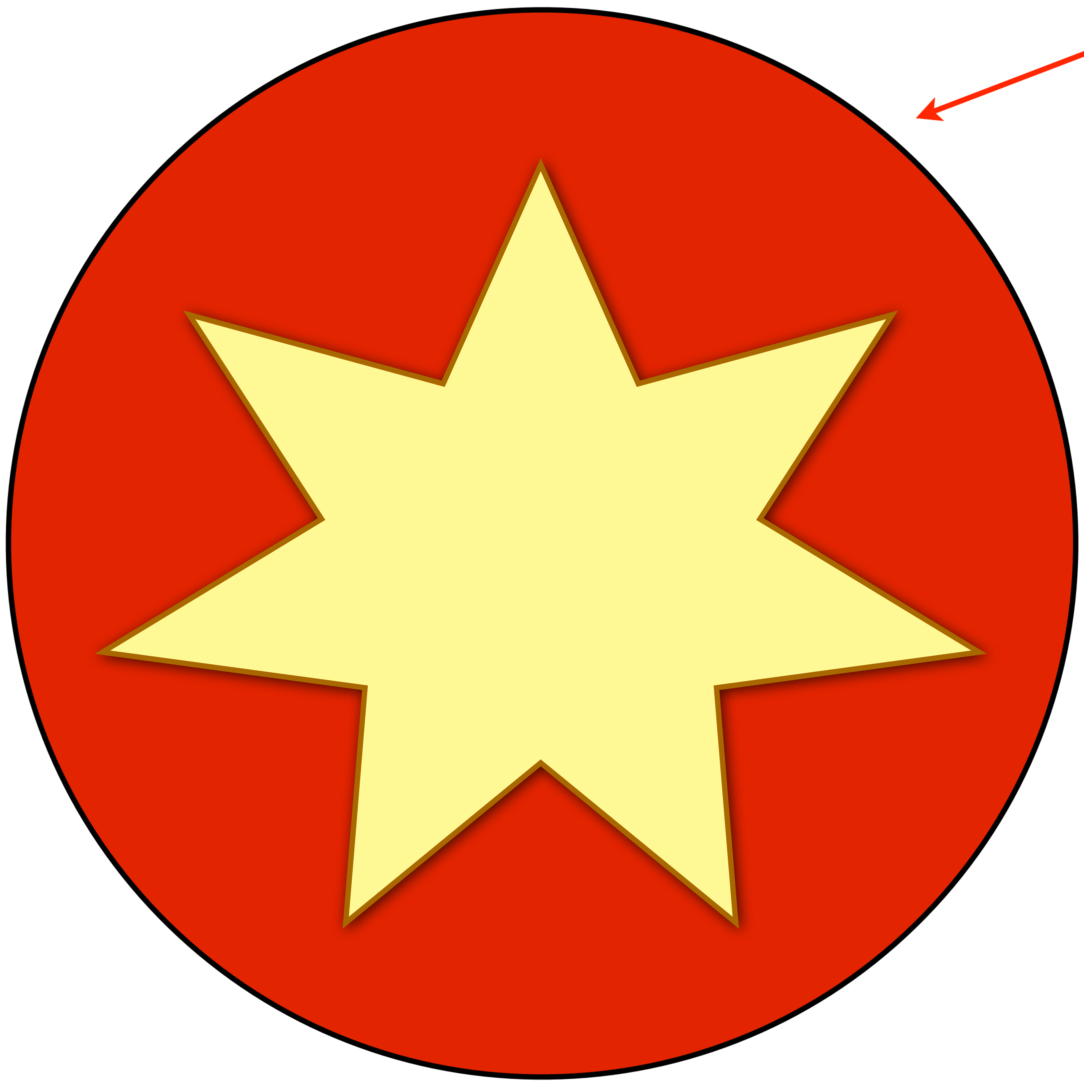

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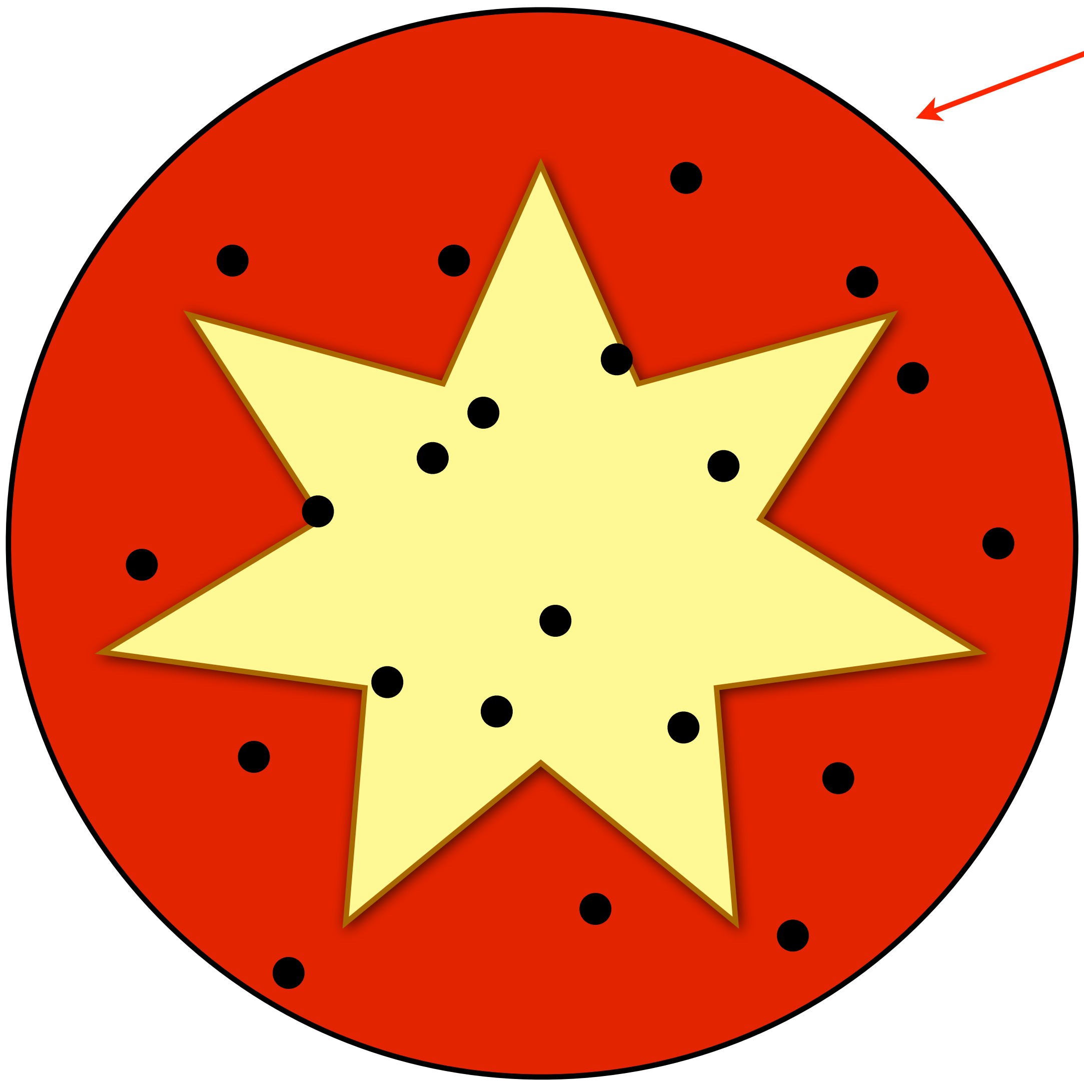

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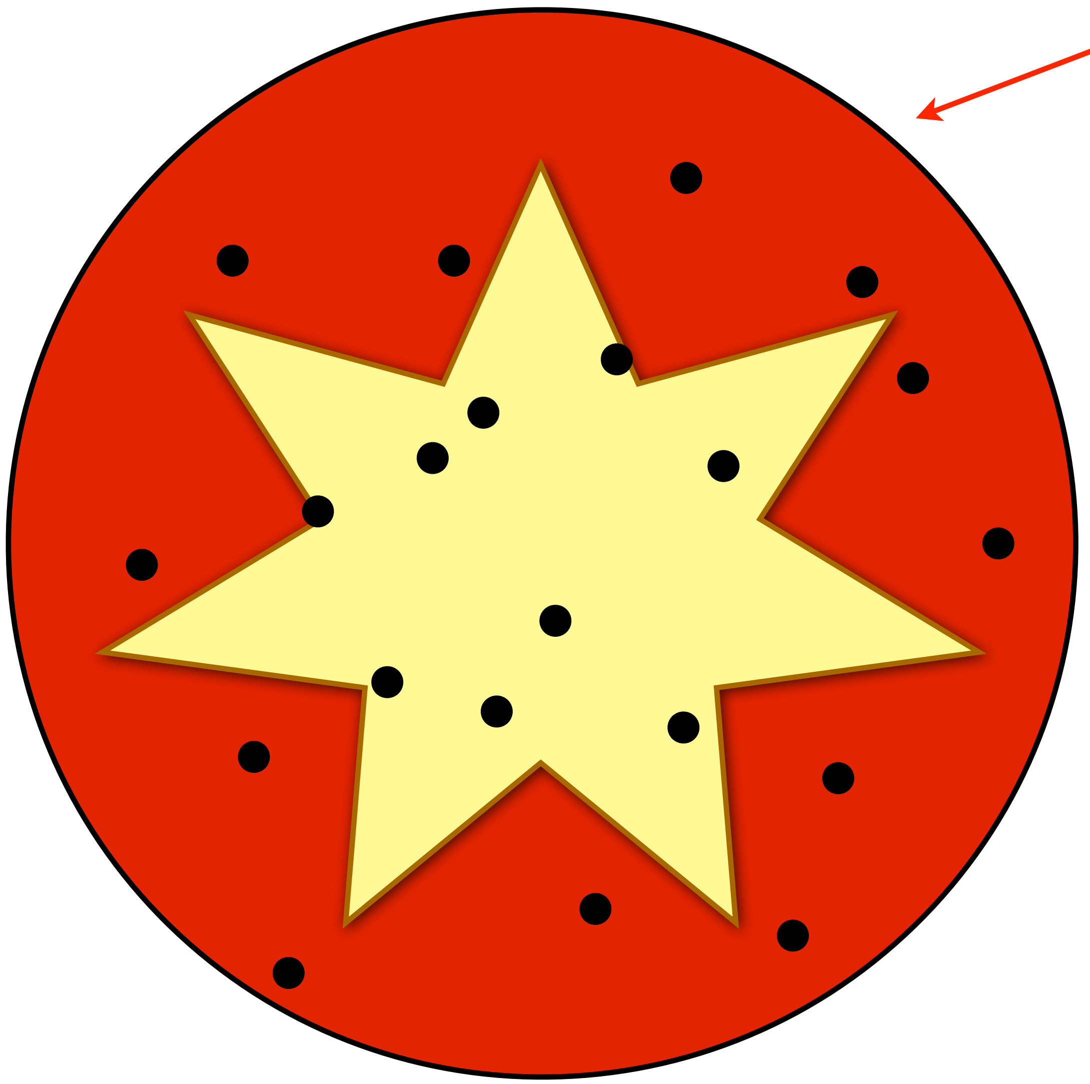

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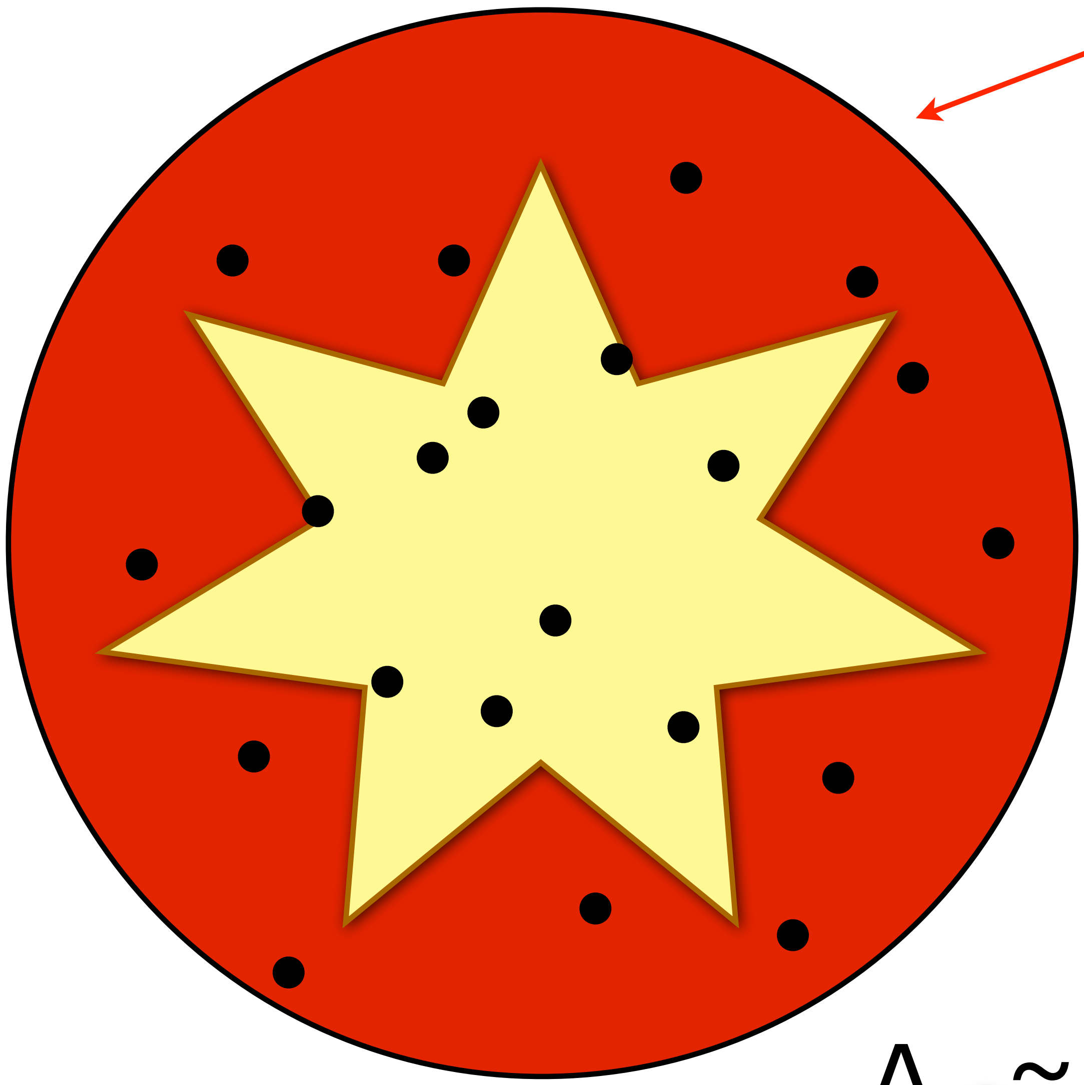

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$$A_{\star} \approx N_{hit}/N_{miss} \times \pi R^2$$

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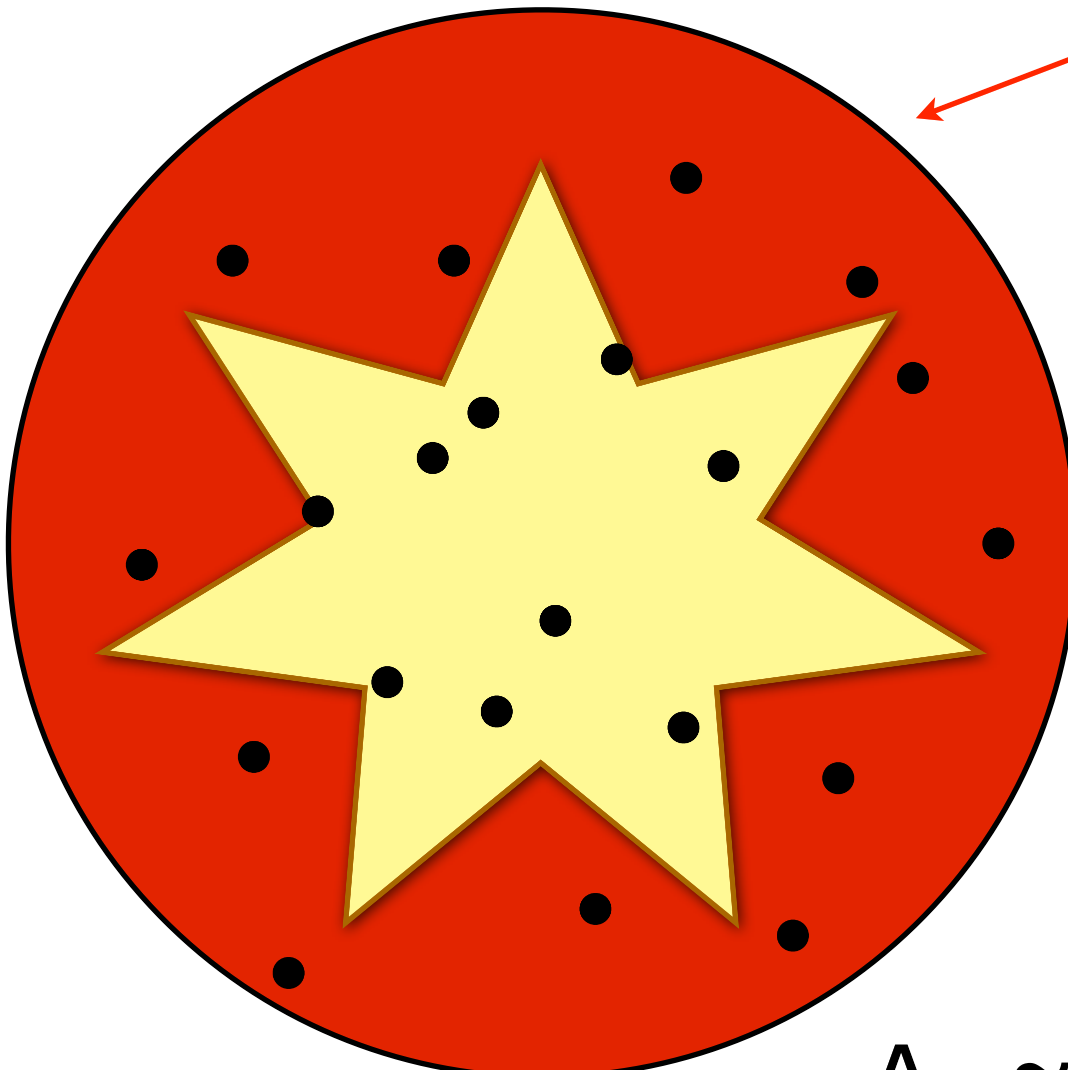

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Earliest Example of MC calculation: Buffon's Needle (1777) to calculate π

G. Leclerc, Comte de Buffon (1707-1788)

$$A_{\star} \approx N_{hit}/N_{miss} \times \pi R^2$$

Random Numbers

I will not tell you how to *write* a Random-number generator (interesting topic & history in its own right)

Instead, I *assume* that you can write a computer code and link to a random-number generator, from a library

- E.g., ROOT includes one that you can use if you like.
- PYTHIA also includes one

From the PYTHIA 8 HTML documentation, under "Random Numbers":

Random numbers R uniformly distributed in 0 < R < 1 are obtained with

```
Pythia8::Rndm::flat();
```

+ Other methods for exp, x*exp, 1D Gauss, 2D Gauss.

Example: Number of Terascale school students who will get hit by a car this week

Complicated Function:

Time-dependent

Traffic density during day, week-days vs week-ends

(i.e., non-trivial time evolution of system)

No two students are the same

Need to compute probability for each and sum

(simulates having several distinct types of "evolvers")

Multiple outcomes:

Hit → keep walking, or go to hospital?

Multiple hits = Product of single hits, or more complicated?

Monte Carlo Approach

Approximate Traffic

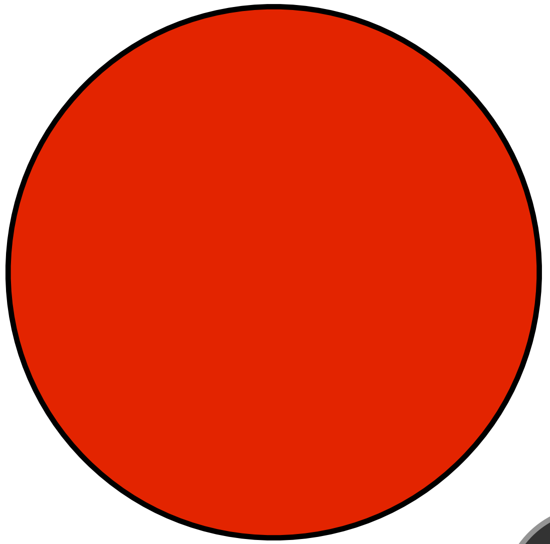
Simple overestimate:

highest recorded density
of most careless drivers,
driving at highest recorded speed
…

Approximate Student

by most completely reckless and accident-prone student (wandering the streets lost in thought after these lectures …)

This extreme guess will be the equivalent of our simple overestimate from before:



Hit Generator

Off we go…

Throw random accidents according to:

$$\sum_{i=1}^{n_{\rm stud}} \alpha_i(x,t)\,\rho_i(x,t)\,\rho_C(x,t)$$

Sum over students

Student-Car hit rate

Density of Student i

Density of Cars

Hit Generator

Off we go…

Throw random accidents according to:

$$R = \int_{t_0}^{t_e} \mathrm{d}t \int_x \mathrm{d}x \sum_{i=1}^{n_{\mathrm{stud}}} \alpha_i(x,t)\, \rho_i(x,t)\, \rho_C(x,t)$$

Sum over students

$\alpha_i(x,t)$: Student-Car hit rate

$\rho_i(x,t)$: Density of Student i

$\rho_C(x,t)$: Density of Cars

t_e : time of accident

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Simple Overestimate

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Too Difficult

↓

Simple Overestimate

$$R = (t_e - t_0)\Delta x \quad \alpha_{\max}\, n_{\mathrm{stud}}\, \rho_{c\max}$$

Hit rate for most accident-prone student — Rush-hour density of cars

Hit Generator

Off we go…

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Too Difficult → Simple Overestimate

$$R = (t_e - t_0)\Delta x \quad \alpha_{\max}\ n_{\mathrm{stud}}\ \rho_{c\max}$$

Hit rate for most accident-prone student — Rush-hour density of cars

(Also generate trial x_e, e.g., uniformly in circle around DESY)

(Also generate trial i; a random student gets hit)

Hit Generator

Accept trial hit (i,x,t) with probability

$$\text{Prob(accept)} = \frac{\alpha_i(x,t)\ \rho_i(x,t)\ \rho_c(x,t)}{\alpha_{\max}\ n_{\text{stud}}\ \rho_{\text{cmax}}}$$

Using the following:

ρ_c : actual density of cars at location x at time t

ρ_i : actual density of student i at location x at time t

α_i : The actual "hit rate" (OK, not really known, but can make one up)

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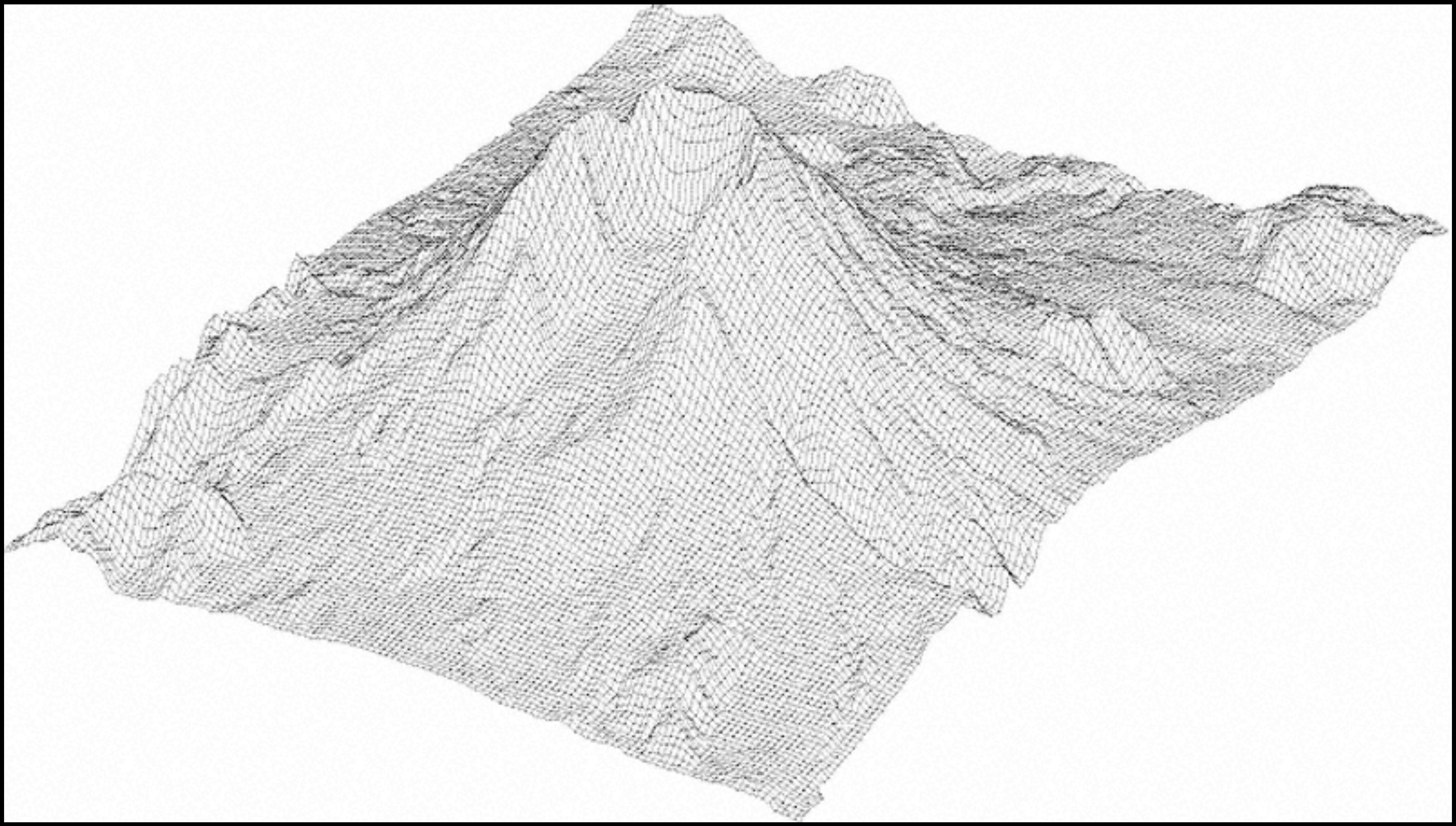
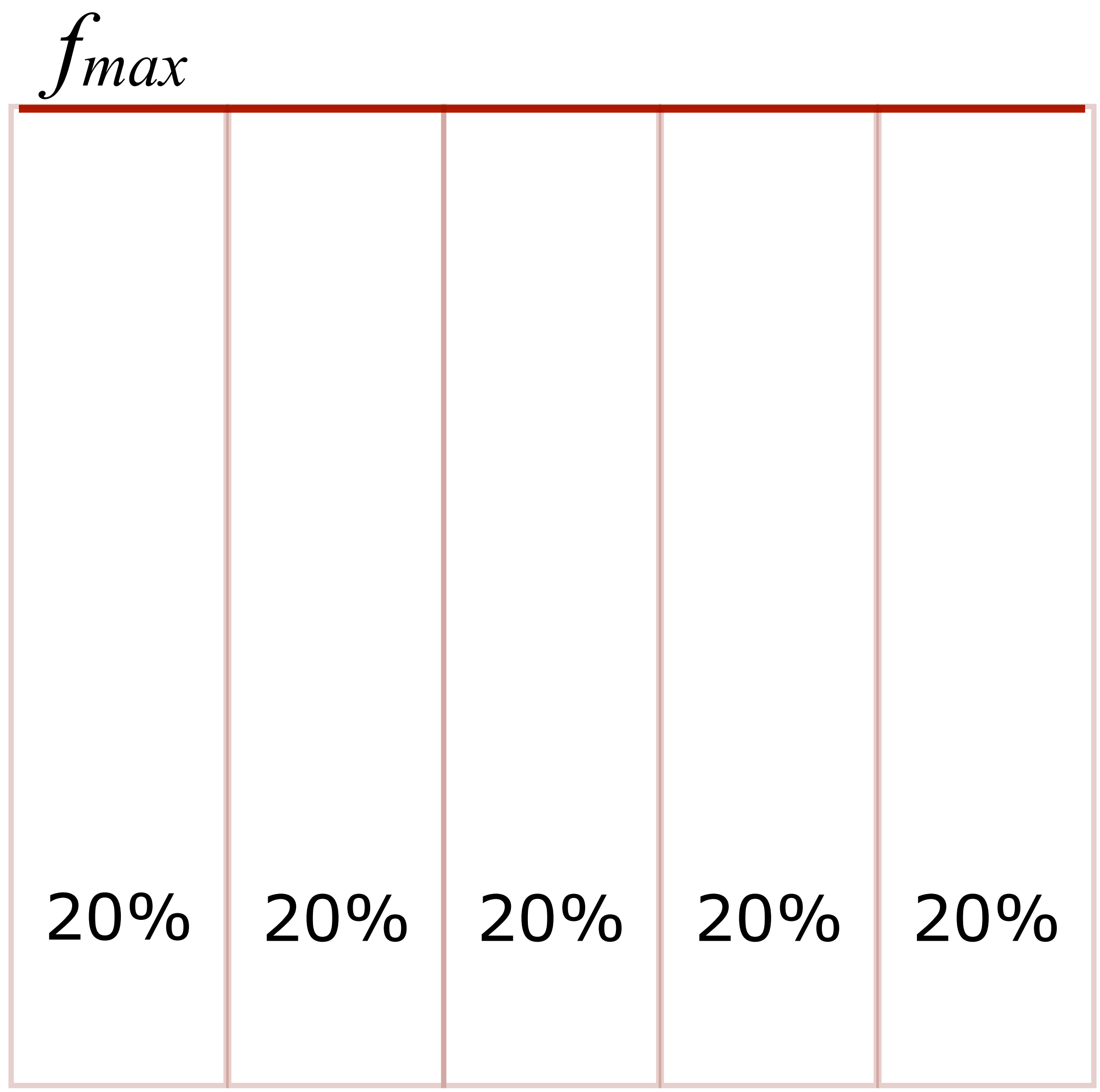
→ True number = number of accepted hits
(note: we didn't really treat multiple hits … → Markov Chain)

Importance Sampling



Peaked Functions



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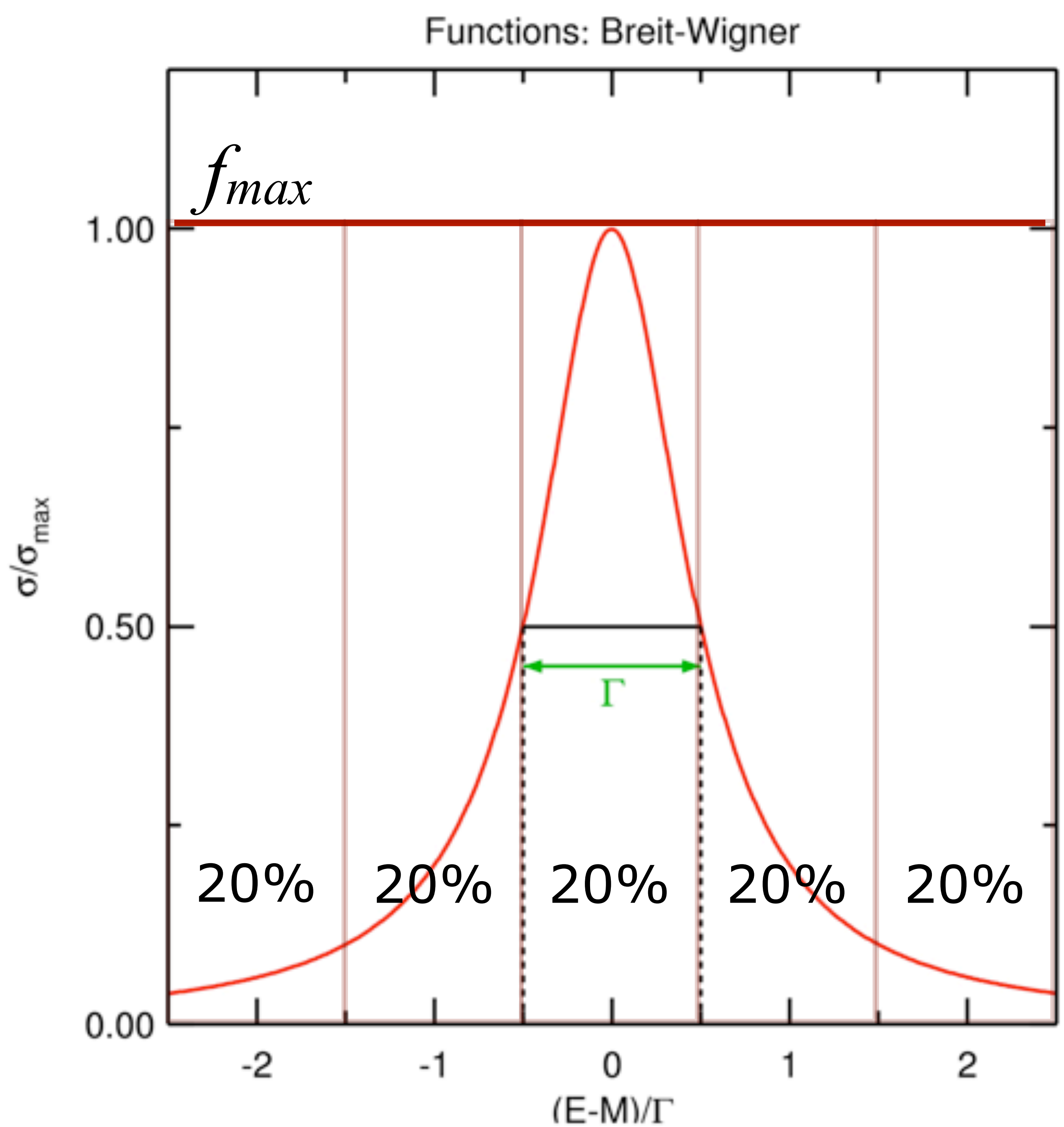

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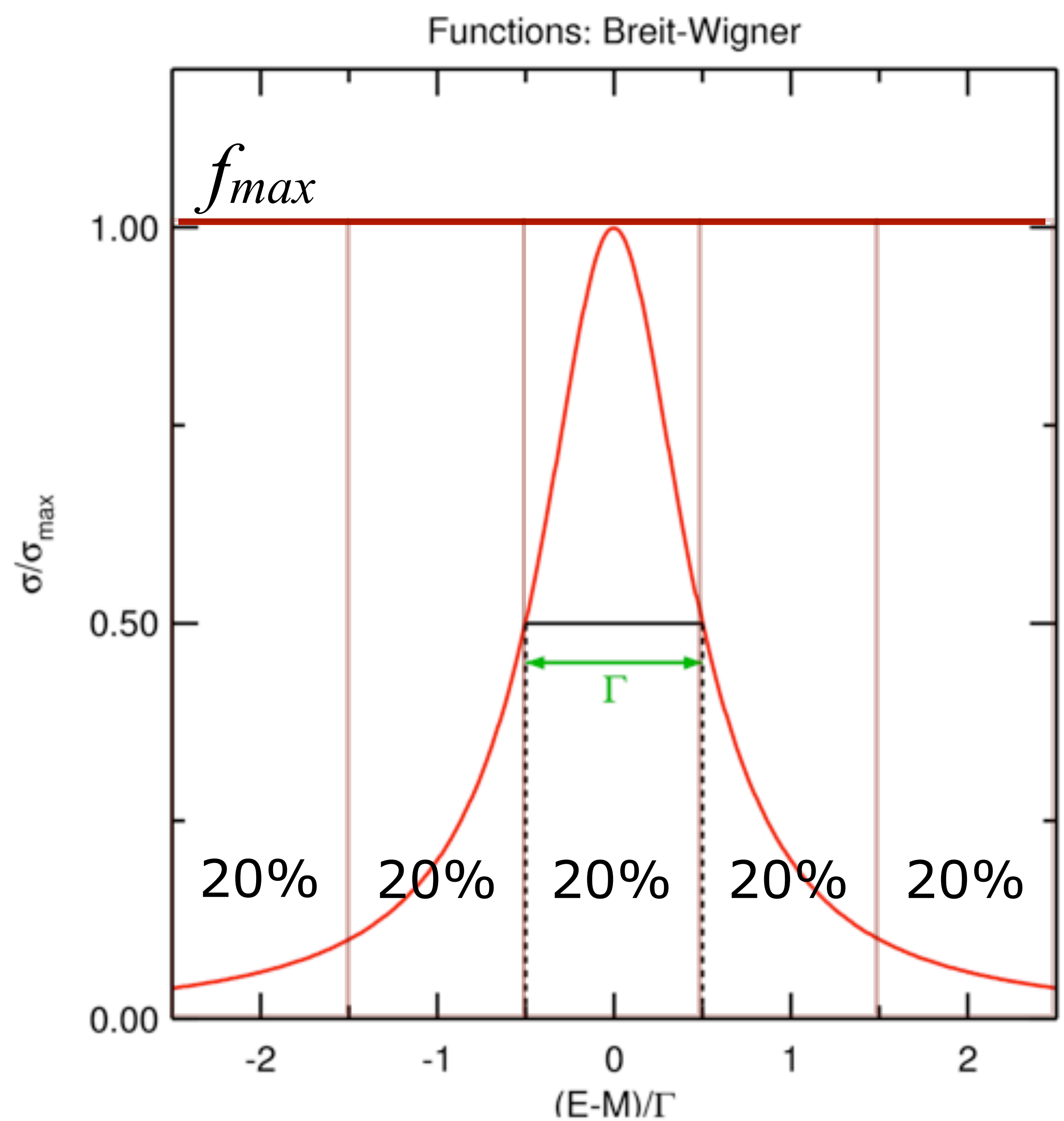
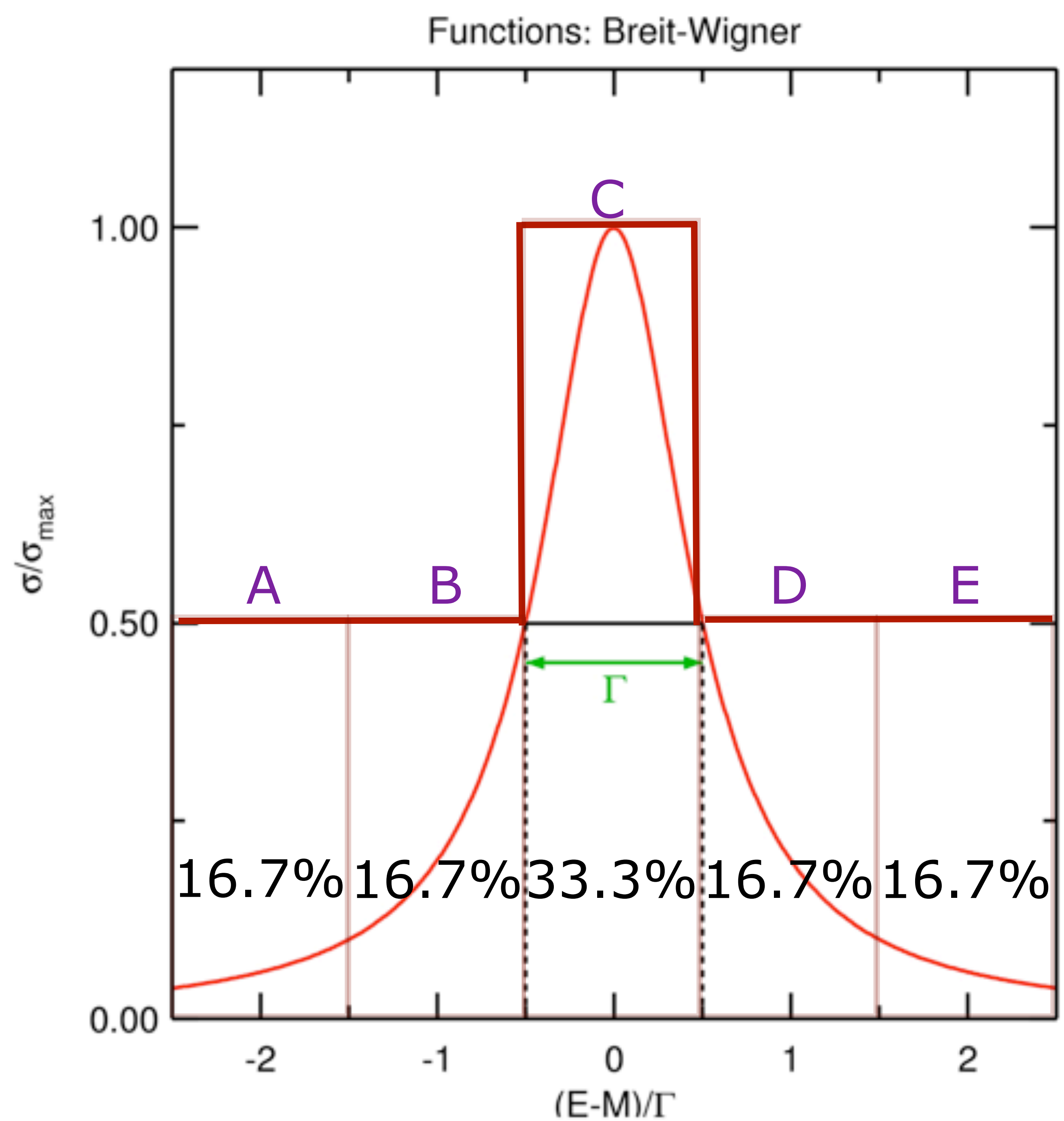


Precision on integral dominated by the points with $f \approx f_{max}$ *(i.e., peak regions)*

→ slow convergence if high, narrow peaks

Stratified Sampling



→ Make it twice as likely to throw points in the peak

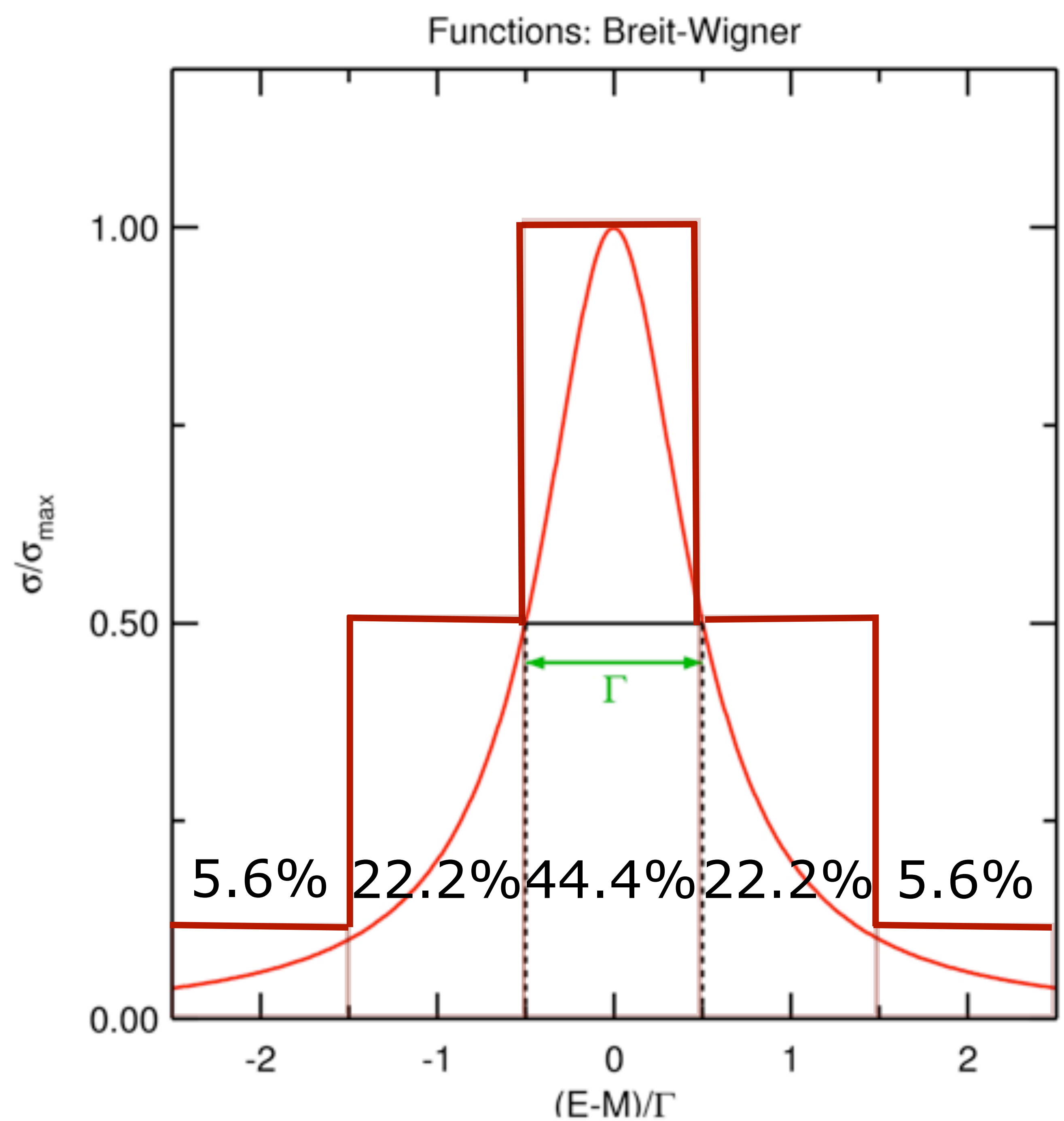
Choose:

For: $6^*R_1 \in$

[0,1] → Region A
[1,2] → Region B
[2,4] → Region C
[4,5] → Region D
[5,6] → Region E

→ faster convergence for same number of function evaluations

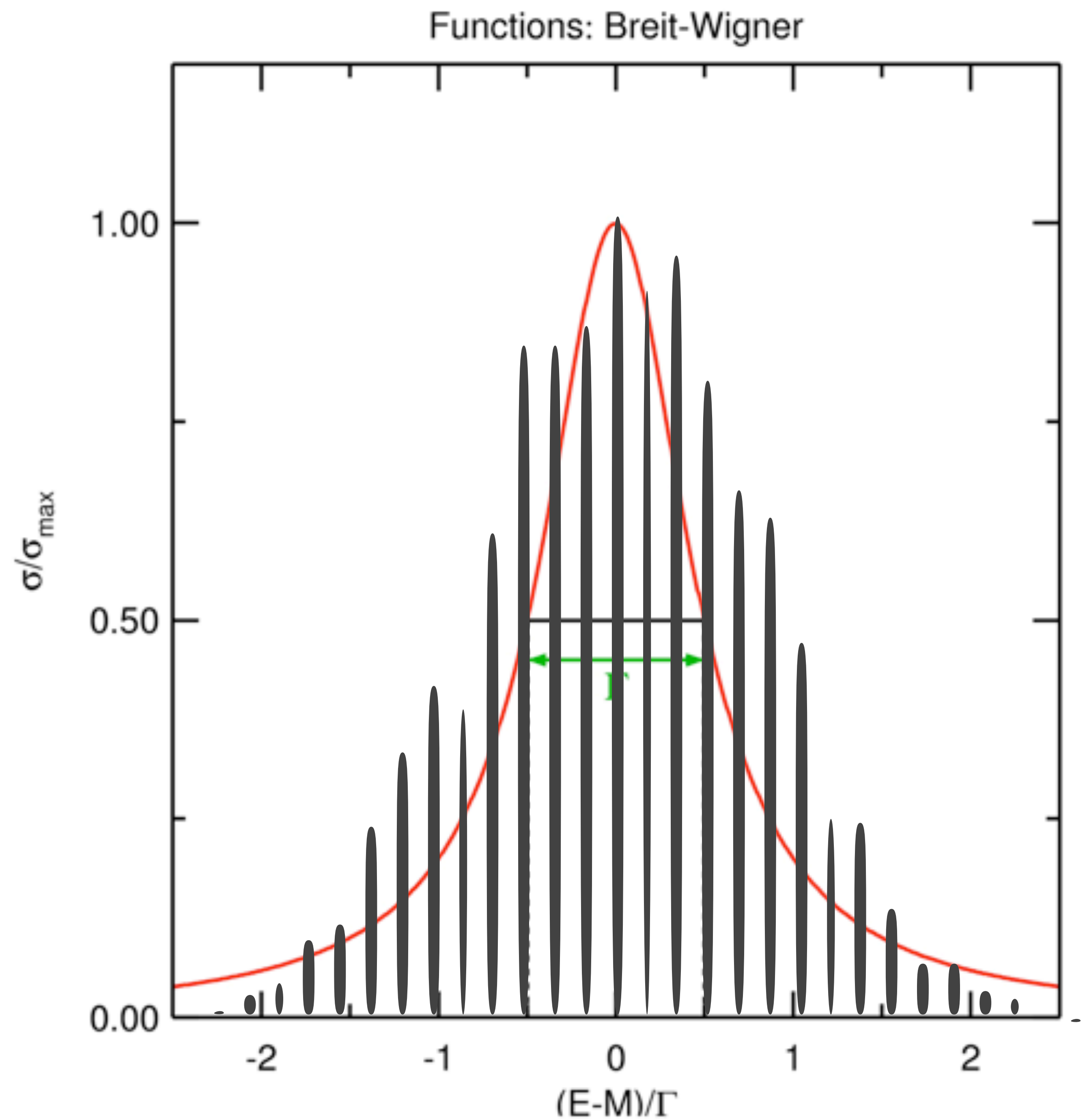
Adaptive Sampling



→ Can even design algorithms to do this automatically as they run (not covered here)

→ Adaptive sampling

Importance Sampling



→ or throw points according to some smooth peaked function for which you have, or can construct, a random number generator (here: Gauss)

E.g., VEGAS algorithm, by G. Lepage

Why does this work?

Why does this work?

1) You are inputting knowledge: obviously need to know where the peaks are to begin with …
(say you know, e.g., the location and width of a resonance)

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2) Stratified sampling increases efficiency by combining fixed-grid methods with the MC method, with further gains from adaptation

3) Importance sampling:

$$\int_a^b f(x)\mathrm{d}x = \int_a^b \frac{f(x)}{g(x)}\mathrm{d}G(x)$$

Effectively does flat MC with changed integration variables

Fast convergence if f(x)/g(x) ≈ 1

The Veto Algorithm

Hit

Miss

How we do Monte Carlo

Take your system

- Set of radioactive nuclei
- Set of hard scattering processes
- Set of resonances that are going to decay
- Set of particles coming into your detector
- Set of cosmic photons traveling across the galaxy
- Set of molecules
- …

How we do Monte Carlo

Take your system

Generate a “trial” (event/decay/interaction/…)

- Not easy to generate random numbers distributed according to exactly the right distribution?
- May have complicated dynamics, interactions …
- → use a simpler “trial” distribution

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May have complicated dynamics, interactions …

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Flat with some stratification

Or importance sample with simple overestimating function (for which you can generate random #s)

How we do Monte Carlo

Take your system

Generate a "trial" (event/decay/interaction/…)

- Accept trial with probability f(x)/g(x)
  - f(x) contains all the complicated dynamics
  - g(x) is the simple trial function
- If accept: replace with new system state
- If reject: keep previous system state

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no dependence on g in final result - only affects convergence rate

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Sounds deceptively simple, but …

with it, you can integrate

arbitrarily complicated functions *(in particular chains of nested functions)*, over arbitrarily complicated regions, in arbitrarily many dimensions …

And keep going: generate next trial …

Summary - Lecture 2

Quantum Scattering Problems are common to many areas of physics: To compute expectation value of observable: integrate over phase space

Complicated functions → Numerical Integration

High Dimensions → Monte Carlo (stochastic) convergence is fastest + Additional power by stratification and/or importance sampling

Additional Bonus → Veto algorithm → direct simulation of arbitrarily complicated reaction chains → "Event Generators"

Recommended Reading

F. James
Monte Carlo Theory and Practice
Rept.Prog.Phys.43 (1980) p.1145

S. Weinzierl
Topical lectures given at the Research School Subatomic physics, Amsterdam, June 2000
Introduction to Monte Carlo Methods
e-Print: hep-ph/0006269

P. Skands
Introduction to QCD (TASI 2012)
arXiv:1207.2389

Conformal QCD in Action

Naively, QCD radiation suppressed by $\alpha_s \approx 0.1$

Truncate at fixed order = LO, NLO, …

But beware the jet-within-a-jet-within-a-jet …

Example:

SUSY pair production at 14 TeV, with $M_{SUSY} \approx 600$ GeV

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LHC - sps1a - m~600 GeV — Plehn, Rainwater, PS PLB645(2007)217

FIXED ORDER pQCD	σ_{tot}[pb]	$\tilde{g}\tilde{g}$	$\tilde{u}_L\tilde{g}$	$\tilde{u}_L\tilde{u}_L^*$	$\tilde{u}_L\tilde{u}_L$	TT
$p_{T,j} >$ 100 GeV	σ_{0j}	4.83	5.65	0.286	0.502	1.30
inclusive **X + 1 "jet"** →	σ_{1j}	2.89	2.74	0.136	0.145	0.73
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σ for X + jets much larger than naive estimate

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(Computed with SUSY-MadGraph)

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σ for 50 GeV jets ≈ larger than total cross section → not under control

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→ More on this in lectures on Jets, Monte Carlo, and Matching

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Scaling Violation

Real QCD isn't conformal

The coupling runs logarithmically with the energy scale

$$Q^2 \frac{\partial \alpha_s}{\partial Q^2} = \beta(\alpha_s) \qquad \beta(\alpha_s) = -\alpha_s^2 (b_0 + b_1 \alpha_s + b_2 \alpha_s^2 + \ldots) \ ,$$

$$b_0 = \frac{11 C_A - 2 n_f}{12\pi} \qquad b_1 = \frac{17 C_A^2 - 5 C_A n_f - 3 C_F n_f}{24\pi^2} = \frac{153 - 19 n_f}{24\pi^2}$$

1-Loop β function coefficient

2-Loop β function coefficient

$$b_2 = \frac{2857 - 5033 n_f + 325 n_f^2}{128\pi^3}$$

$$b_3 = \text{known}$$

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Asymptotic freedom in the ultraviolet

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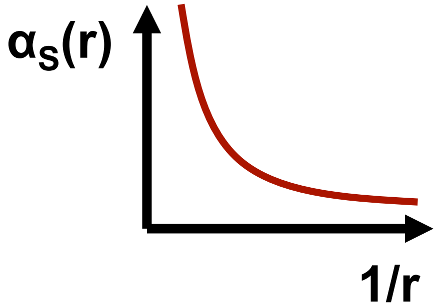
Asymptotic freedom in the ultraviolet

Confinement (IR slavery?) in the infrared

Asymptotic Freedom

“What this year's Laureates discovered was something that, at first sight, seemed completely contradictory. The interpretation of their mathematical result was that the closer the quarks are to each other, the *weaker* is the 'colour charge'. When the quarks are really close to each other, the force is so weak that they behave almost as free particles. This phenomenon is called ‘asymptotic freedom’. The converse is true when the quarks move apart: the force becomes stronger when the distance increases.”



Nobelprize.org
The Official Web Site of the Nobel Prize

The Nobel Prize in Physics 2004
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David J. Gross · H. David Politzer · Frank Wilczek

The Nobel Prize in Physics 2004 was awarded jointly to David J. Gross, H. David Politzer and Frank Wilczek *"for the discovery of asymptotic freedom in the theory of the strong interaction"*.

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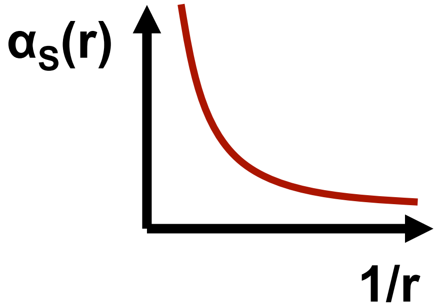


*1 The force still goes to ∞ as $r \to 0$ (Coulomb potential), just less slowly

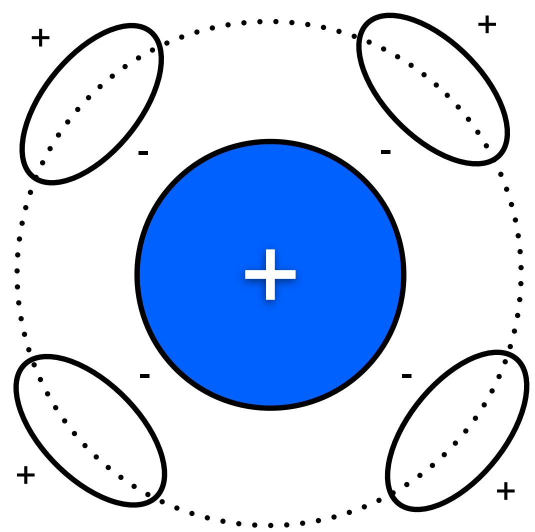
*2 The potential grows linearly as $r \to \infty$, so the force actually becomes constant (even this is only true in "quenched" QCD. In real QCD, the force eventually vanishes for $r \gg 1$ fm)

Asymptotic Freedom

QED:

Vacuum polarization

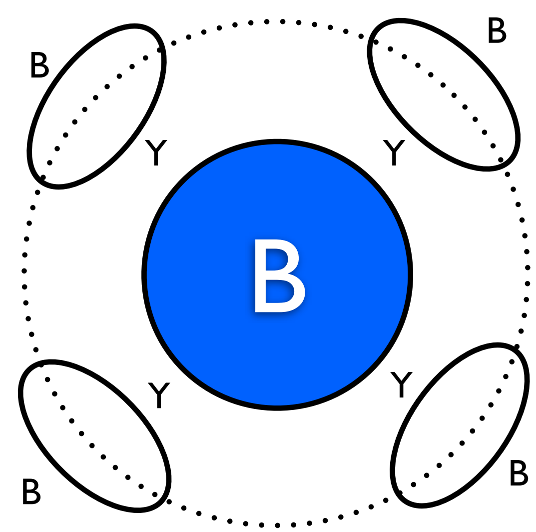
→ Charge screening



QCD:

Quark Loops

→ Also charge screening



But only dominant if > 16 flavors!

Asymptotic Freedom

$$b_0 = \frac{11C_A - 2n_f}{12\pi}$$

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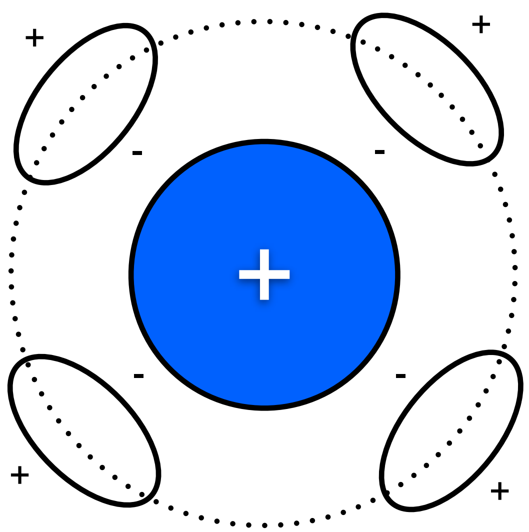
Vacuum polarization

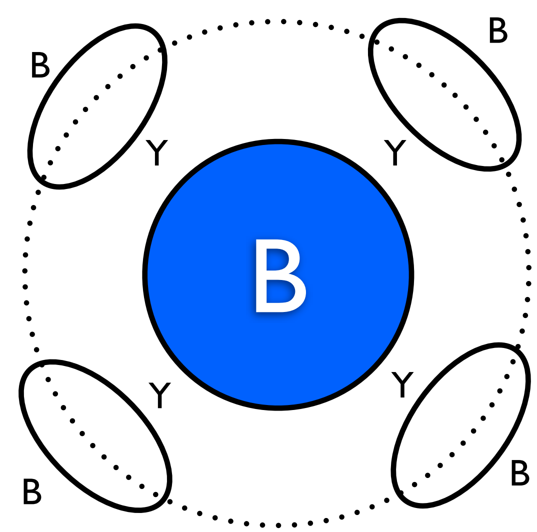
→ Charge screening

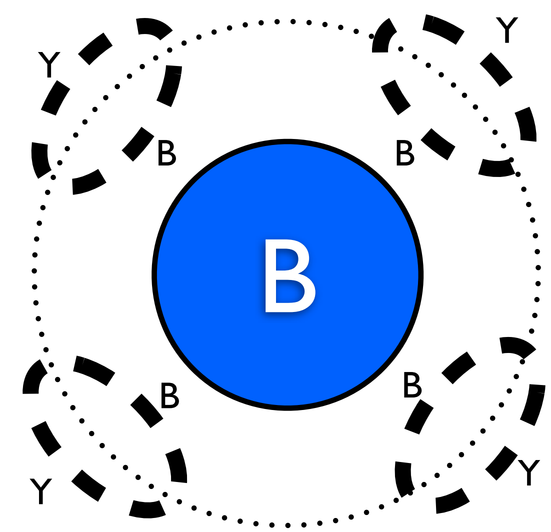


QCD:

Gluon Loops

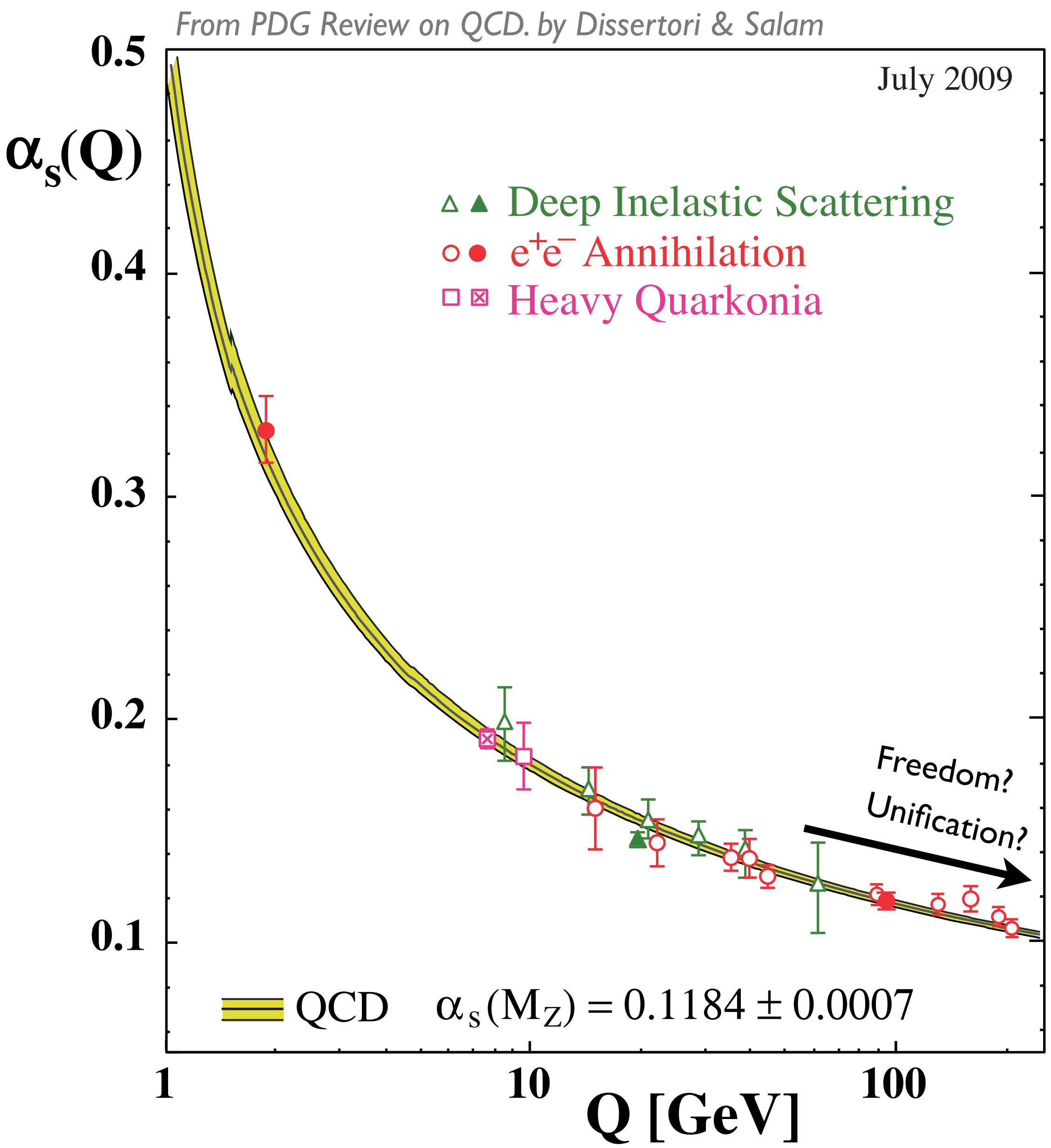
Dominate if ≤ 16 flavors





Spin-1 → Opposite Sign

UV and IR



At low scales

Coupling $\alpha_s(Q)$ actually runs rather fast with Q

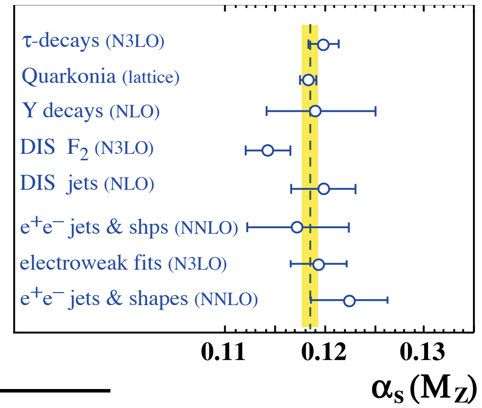
Perturbative solution diverges at a scale Λ_{QCD} somewhere below

$\approx$ 1 GeV

So, to specify the strength of the strong force, we usually give the value of α_s at a unique reference scale that everyone agrees on: M_Z

The Fundamental Parameter(s)

From PDG Review on QCD. by Dissertori & Salam



QCD has one fundamental parameter

$$\alpha_s(m_Z)^{\overline{\mathrm{MS}}}$$

$$\alpha_s(Q^2) = \alpha_s(m_Z^2) \frac{1}{1 + b_0\, \alpha_s(m_Z) \ln \frac{Q^2}{m_Z^2} + \mathcal{O}(\alpha_s^2)}$$

$$b_0 = \frac{11 N_C - 2 n_f}{12\pi}$$

… + n_f and quark masses

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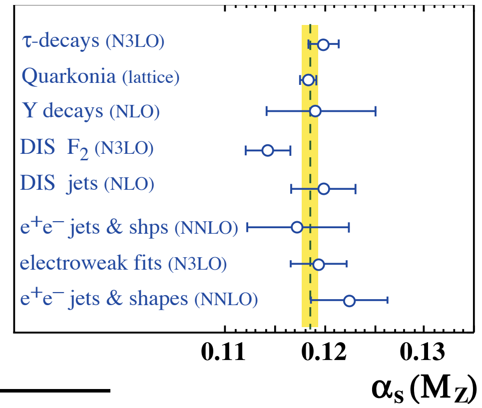

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… and its sibling

$\Lambda_{\mathrm{QCD}}^{(n_f)\overline{\mathrm{MS}}}$

$$\alpha_s(Q^2) = \frac{1}{b_0 \;\ln \frac{Q^2}{\Lambda^2}}$$

(depends on n_f, scheme, and # of loops)

$$\Lambda \sim 200\ \mathrm{MeV}$$

… + n_f and quark masses

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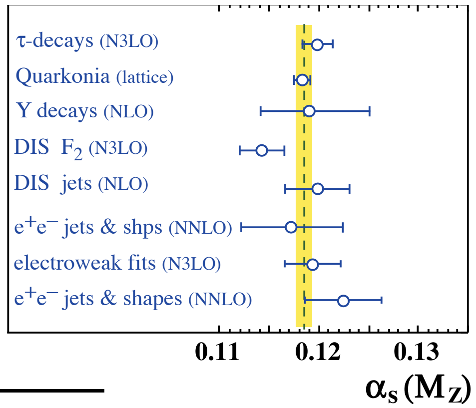

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… And all its cousins

$\Lambda^{(3)}$ $\Lambda^{(4)}$ $\Lambda^{(5)}$ Λ_{CMW} Λ_{FSR} Λ_{ISR} Λ_{MPI}, …

… + n_f and quark masses

Uncalculated Orders

Naively $O(\alpha_s)$ - True in e^+e^- !

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Generally larger in hadron collisions

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Uncalculated Orders

Naively O(α_s) - True in e^+e^- !

$$\sigma_{\mathrm{NLO}}(e^+e^- \to q\bar{q}) = \sigma_{\mathrm{LO}}(e^+e^- \to q\bar{q}) \left(1 + \frac{\alpha_s(E_{\mathrm{CM}})}{\pi} + \mathcal{O}(\alpha_s^2)\right)$$

Generally larger in hadron collisions

Typical "K" factor in pp (= $\sigma_{\mathrm{NLO}}/\sigma_{\mathrm{LO}}$) ≈ 1.5 ± 0.5

Why is this? Many pseudoscientific explanations

- Explosion of # of diagrams ($n_{\mathrm{Diagrams}} \approx n!$)
- New initial states contributing at higher orders (E.g., $gq \to Zq$)
- Inclusion of low-x (non-DGLAP) enhancements
- Bad (high) scale choices at Lower Orders, …

Theirs not to reason why // Theirs but to do and die

Tennyson, The Charge of the Light Brigade

Changing the scale(s)

Why scale variation ~ uncertainty?

Scale dependence of calculated orders must be canceled by contribution from uncalculated ones (+ non-pert)

$$\alpha_s(Q^2) = \alpha_s(m_Z^2)\frac{1}{1 + b_0\ \alpha_s(m_Z)\ln\frac{Q^2}{m_Z^2} + \mathcal{O}(\alpha_s^2)}$$

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→ $$\left(\alpha_s(Q'^2) - \alpha_s(Q^2)\right)|M|^2 = \alpha_s^2(Q^2)|M|^2 + \ldots$$

→ Generates terms of higher order, but proportional to what you already have ($|M|^2$)→ a first naive* way to estimate uncertainty

*warning: some theorists believe it is the only way … but be agnostic! There are other things than scale dependence …

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Complicated final states

Intrinsically Multi-Scale problems

with Many powers of α_s

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$$\alpha_s(p_{\perp 1})\alpha_s(p_{\perp 2})\alpha_s(p_{\perp 3}) \sim \alpha_s^3\left(\langle p_\perp^2 \rangle_{\mathrm{geom}}\right)$$

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Dangers

$p_{\perp 1}$= 50 GeV
$p_{\perp 2}$= 50 GeV
$p_{\perp 3}$= 50 GeV

Complicated final states

Intrinsically Multi-Scale problems

with Many powers of α_s

E.g., W + 3 jets in pp

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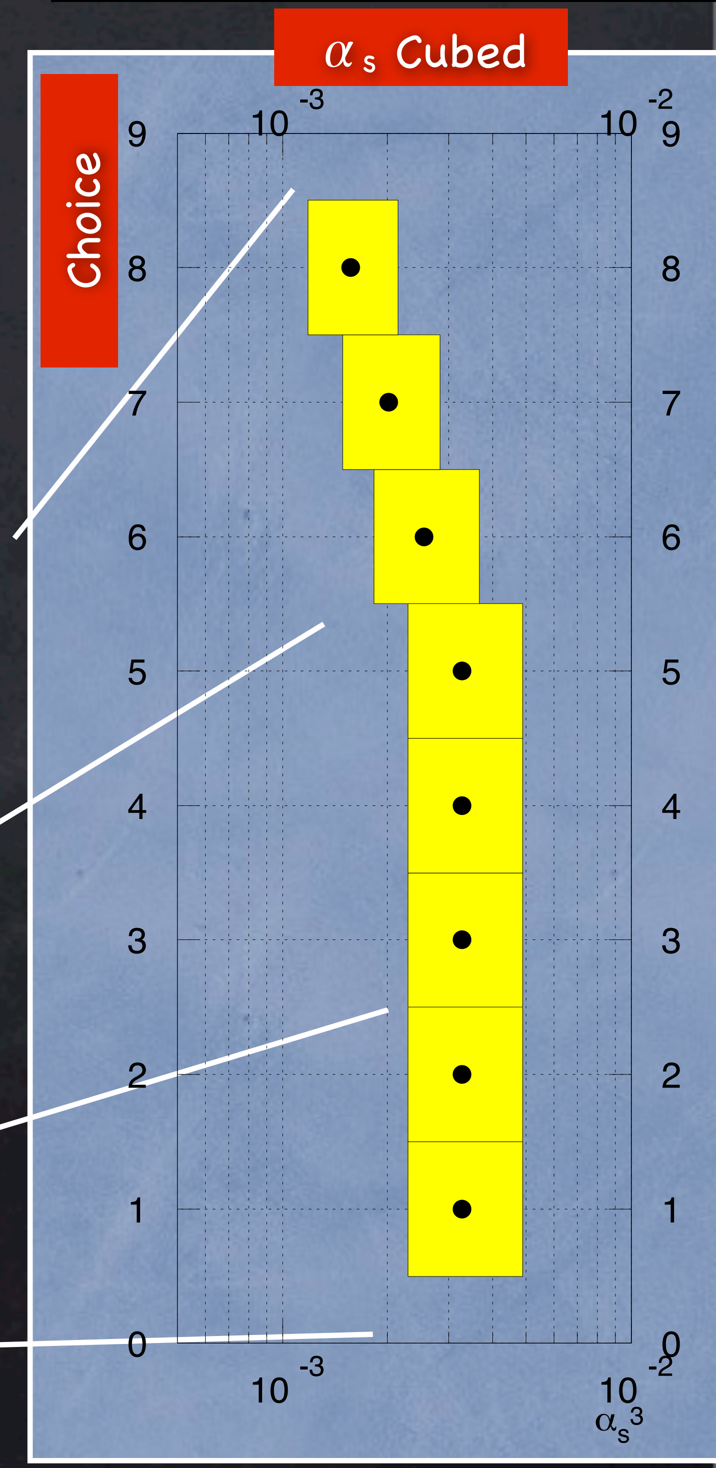

Dangers

$p_{\perp 1}$= 500 GeV
$p_{\perp 2}$= 100 GeV
$p_{\perp 3}$= 30 GeV

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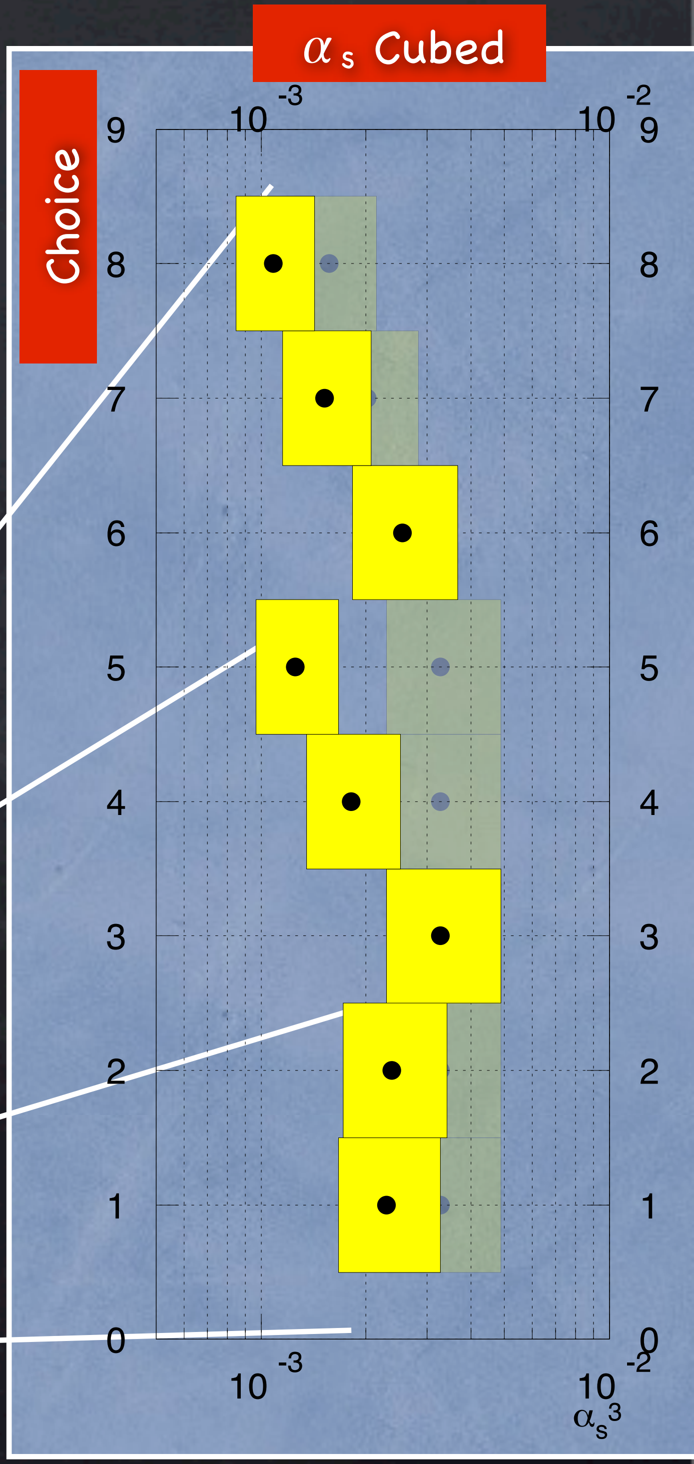

If you have multiple QCD scales

→ variation of μ_R by factor 2 in each direction not good enough! (nor is × 3, nor × 4)

Need to vary also functional dependence on each scale!