# QFT Beyond Fixed Order

## Introduction to Bremsstrahlung and Jets

### 1. Radiation from Accelerated Charges

*Soft Bremsstrahlung in Classical E&M, and in QED.* ***The dipole factor*** *& coherence.*

### 2. Infrared Singularities and Infrared Safety

*IR Poles & Sudakov Logarithms.* ***Probabilities > 1.***

*Summing over degenerate quantum states (KLN theorem).* ***IRC Safety.***

### 3. QCD as a Weakly Coupled Conformal Field Theory

*The* ***emission*** *probability; Double-Logarithmic Approximation*

*The* ***no-emission*** *probability; Sudakov Factor; exponentiation; example:* ***jet mass****.*

### 4. Parton Showers

*Differential evolution kernels; evolution scale; unitarity and detailed balance.*

*Sampling the Sudakov; perturbation theory as a Monte Carlo Markov Chain.*

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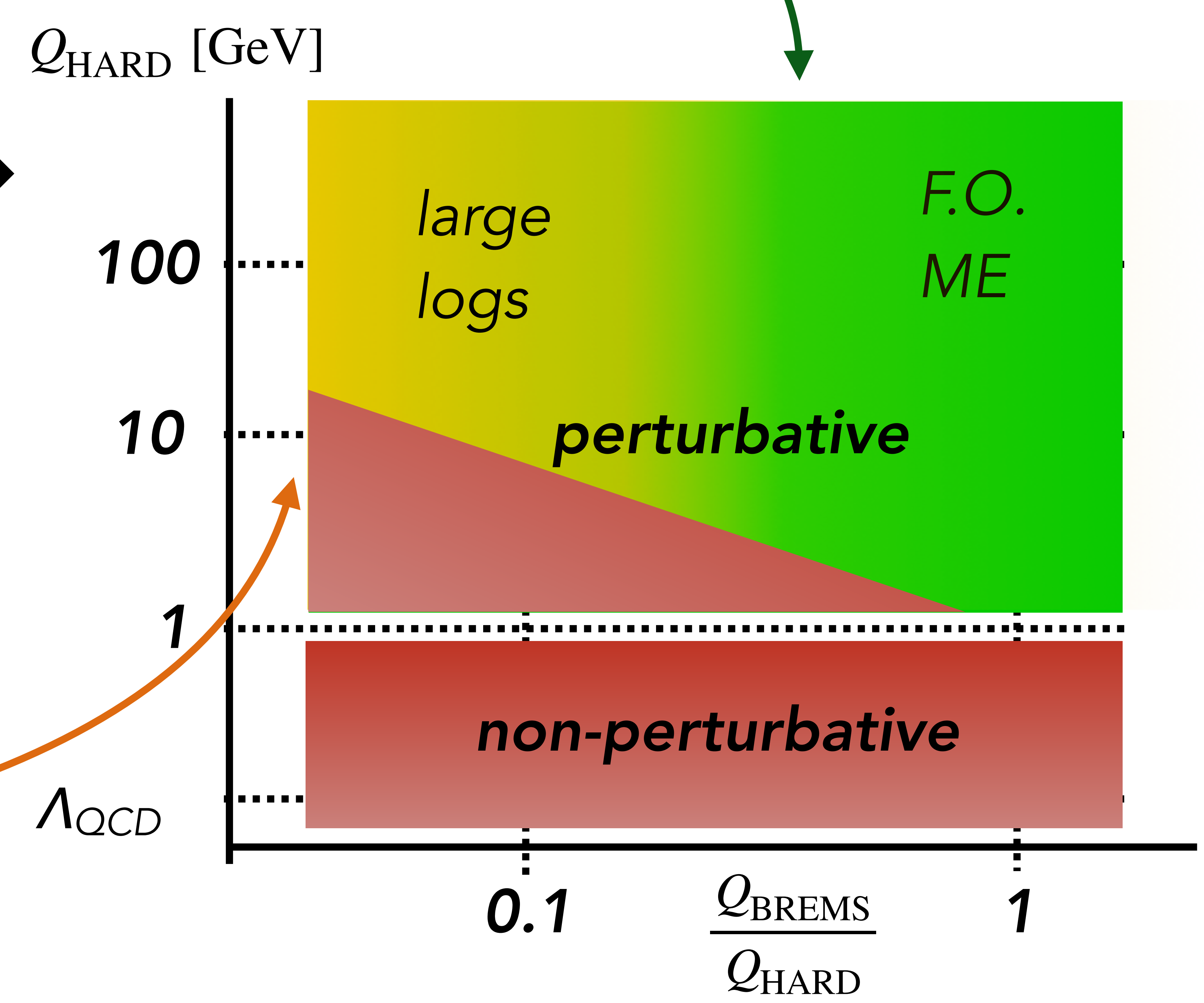
**Fixed-Order perturbative QCD requires Large scales ($\alpha_s$ small enough to be perturbative → high-scale processes)**

**Fixed-Order QCD** also requires **No hierarchies:**

**Bremsstrahlung propagators** $\propto 1/Q^2$
integrated over phase space $\propto dQ^2$ →
**logarithms**

$$\alpha_s^n \ln^m \left(Q^2_{\mathrm{Hard}}/Q^2_{\mathrm{Brems}}\right) \quad ; \; m \leq 2n$$

→ cannot truncate at any fixed order $n$ if upper and lower integration limits are **hierarchically different**




**For observables that involve scale hierarchies: need methods beyond fixed order**

# Example: SUSY + Jets at LHC

## Naively, QCD radiation suppressed by $\alpha_s \approx 0.1$

→ Truncate at fixed order = LO, NLO, …

But beware the jet-within-a-jet-within-a-jet …

$\Longrightarrow$ 100 GeV can be "soft" at the LHC

## Example: SUSY pair production at LHC$_{14}$, with $M_{SUSY} \approx 600$ GeV

LHC - sps1a - m~600 GeV

Plehn, Rainwater, PS PLB645(2007)217

| FIXED ORDER pQCD | $\sigma_{tot}$ [pb] | $\tilde{g}\tilde{g}$ | $\tilde{u}_L\tilde{g}$ | $\tilde{u}_L\tilde{u}_L^*$ | $\tilde{u}_L\tilde{u}_L$ | $TT$ |
|---|---|---|---|---|---|---|
| $p_{T,j} >$ 100 GeV | $\sigma_{0j}$ | 4.83 | 5.65 | 0.286 | 0.502 | 1.30 |
| inclusive **X + 1 "jet"** → | $\sigma_{1j}$ | 2.89 | 2.74 | 0.136 | 0.145 | 0.73 |
| inclusive **X + 2 "jets"** → | $\sigma_{2j}$ | 1.09 | 0.85 | 0.049 | 0.039 | 0.26 |
| $p_{T,j} >$ 50 GeV | $\sigma_{0j}$ | 4.83 | 5.65 | 0.286 | 0.502 | 1.30 |
| | $\sigma_{1j}$ | 5.90 | 5.37 | 0.283 | 0.285 | 1.50 |
| | $\sigma_{2j}$ | 4.17 | 3.18 | 0.179 | 0.117 | 1.21 |

(Computed with SUSY-MadGraph)

σ for X + jets much larger than naive factor-$\alpha_s$ estimate

σ for 50 GeV jets ≈ larger than total cross section → what is going on?

All the scales are high, $Q \gg 1$ GeV, so perturbation theory **should** be OK

# Harder Processes are accompanied by Harder Jets

## Hard processes "kick off" showers of successively softer radiation

Fractal structure: if you look at $Q_{JET}/Q_{HARD} \ll 1$, you **will** resolve substructure.

So it's **not** like you can put a cut at X (e.g., 50, or even 100) GeV and say: "Ok, now fixed-order matrix elements will be OK"

## Extra radiation:

Will generate **corrections to your kinematics**

**Extra jets** from bremsstrahlung can be important **combinatorial background** especially if you are looking for decay jets of similar $p_T$ scales (often, $\Delta M \ll M$)

Is an unavoidable aspect of the **quantum description of quarks and gluons** (no such thing as a "bare" quark or gluon; they depend on how you look at them)

**This is what parton showers are for**

# Evolution Equations

## What we need is a differential equation

Boundary condition: a few partons defined at a high scale ($Q_F$)

Then evolves (or "runs") that parton system down to a low scale (the hadronization cutoff ~ 1 GeV) → It's an evolution equation in $Q_F$

## Close analogue: nuclear decay

Evolve an unstable nucleus. Check if it decays + follow chains of decays.

**Decay constant**

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = c_N$$

**Probability to remain undecayed in the time interval [$t_1$,$t_2$]**

$$\Delta(t_1,t_2) = \exp\left(-\int_{t_1}^{t_2} c_N \, \mathrm{d}t\right) = \exp\left(-c_N \, \Delta t\right)$$

$$= 1 - c_N \Delta t + \mathcal{O}(c_N^2)$$

**Physical decay rate per unit time**

$$\frac{\mathrm{d}P_{\mathrm{res}}(t)}{\mathrm{d}t} = \frac{-\mathrm{d}\Delta}{\mathrm{d}t} = c_N \, \Delta(t_1,t)$$

(respects that each of the original nuclei can only decay if not decayed already)

$\Delta(t_1,t_2)$ : "Sudakov Factor"

# The Sudakov Factor

## In nuclear decay, the Sudakov factor counts:

What fraction of nuclei remain undecayed after a time $t$:

Probability to remain undecayed in the time interval $[t_1,t_2]$

$$\Delta(t_1, t_2) = \exp\left(-\int_{t_1}^{t_2} c_N \, \mathrm{d}t\right) = \exp\left(-c_N \, \Delta t\right)$$

## The Sudakov factor for a parton system "counts":

The probability that the parton system doesn't evolve (branch) when we run the factorization scale (~1/time) from a high to a low scale (i.e., that there is no state change)

Evolution probability per unit "time"

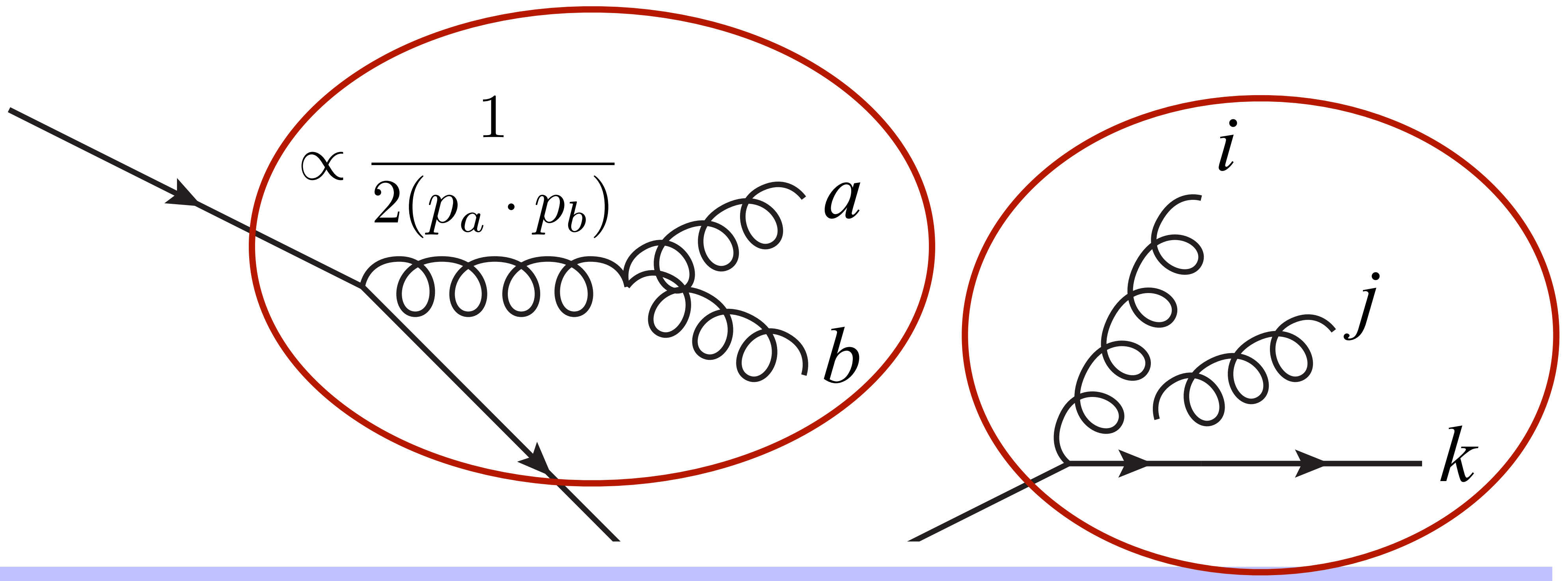
$$\frac{\mathrm{d}P_{\mathrm{res}}(t)}{\mathrm{d}t} = \frac{-\mathrm{d}\Delta}{\mathrm{d}t} = c_N \, \Delta(t_1, t)$$

1. Replace $c_N$ by proper QCD / QED branching densities (e.g., our dipole factor)
2. Replace $t$ by proper definition of "shower evolution scale" ~ resolution scale.
3. Cast as Markov Chain Monte Carlo: sample $t$ steps stochastically + iterative state changes.

# 1. What are the Shower Evolution Kernels?

Most bremsstrahlung is driven by **divergent propagators** → simple universal structure, independent of process details

Bremsstrahlung




**Amplitudes *factorise* in singular limits:**

**Partons ab → "collinear"**

$P(z)$ = **DGLAP** splitting kernels, with $z$ = energy fraction = $E_a/(E_a+E_b)$

$$|\mathcal{M}_{F+1}(\ldots,a,b,\ldots)|^2 \stackrel{a||b}{\rightarrow} g_s^2 \mathcal{C} \frac{P(z)}{2(p_a \cdot p_b)} |\mathcal{M}_F(\ldots,a+b,\ldots)|^2$$

**Coherence** → Parton $j$ really emitted by $(i,k)$ colour dipole: **eikonal**

**Gluon j → "soft":**

$$|\mathcal{M}_{F+1}(\ldots,i,j,k\ldots)|^2 \stackrel{j_g\rightarrow 0}{\rightarrow} g_s^2 \mathcal{C} \frac{(p_i \cdot p_k)}{(p_i \cdot p_j)(p_j \cdot p_k)} |\mathcal{M}_F(\ldots,i,k,\ldots)|^2$$

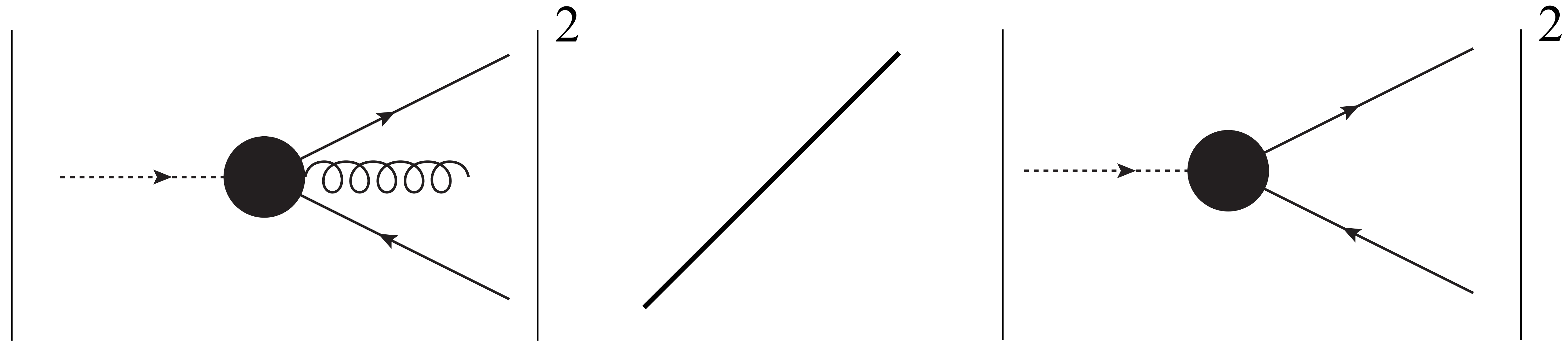
Apply this **many times** for successively softer / more collinear emissions ➜ **QCD fractal**

+ scaling ***violation***: $g_s^2 \rightarrow 4\pi\alpha_s(Q^2)$

# (Types of Showers)

Factorisation of (squared) amplitudes in IR singular limits

(leading colour)




Full ME (modulo nonsingular terms)

**DGLAP**

*ij*-collinear limit

*jk*-collinear limit

$$\frac{P_{q\to qg}(z_i)}{s_{qg}} + \frac{P_{q\to qg}(z_k)}{s_{g\bar{q}}}$$

One term for each **parton**

**Not** a priori coherent.

\+ **Angular ordering** restores azimuthally averaged eikonal

**Antenna**

$$\frac{2s_{q\bar{q}}}{s_{qg}s_{g\bar{q}}} + \frac{1}{s}\left(\frac{s_{g\bar{q}}}{s_{qg}} + \frac{s_{qg}}{s_{g\bar{q}}}\right)$$

eikonal term

collinear terms

partitioning of eikonal

**One** term for each colour connection

**Coherent** by construction

**Dipole (CS/Partitioned)**

$$\frac{\mathcal{K}_{qg,\bar{q}}(z_q)}{s_{qg}} + \frac{\mathcal{K}_{\bar{q}g,q}(z_{\bar{q}})}{s_{g\bar{q}}}$$

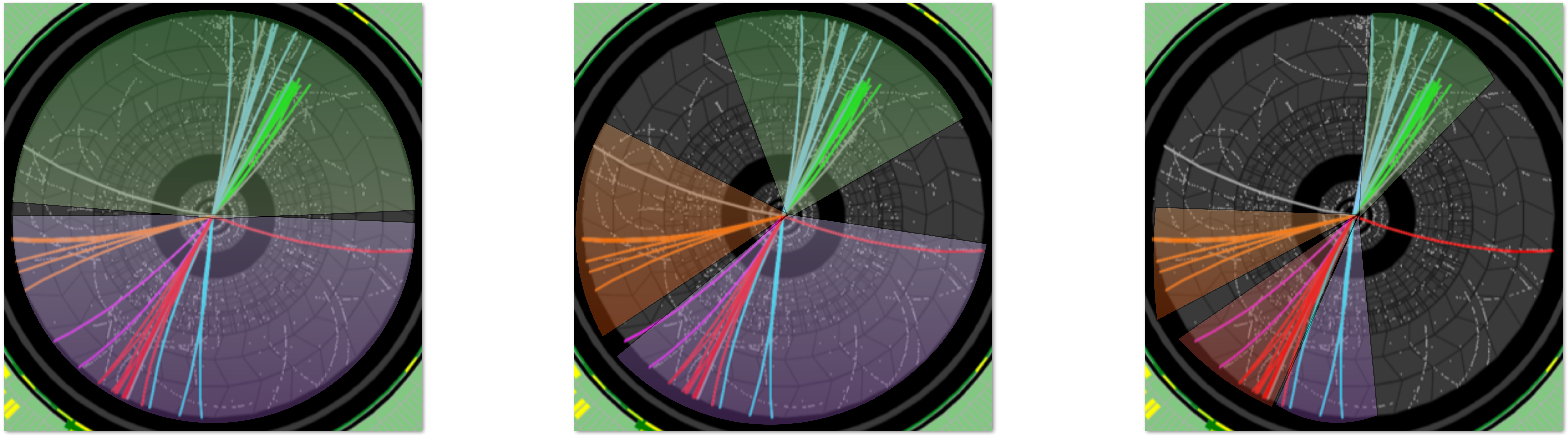
**Two** terms for each colour connection

**Coherent** by construction

**Note:** this is (intentionally) oversimplified. Many subtleties (recoil strategies, gluon parents, initial-state partons, and mass terms) not shown.

## We are working in momentum space

Resolution variable should be an energy scale $Q \sim 1/t$



In the example with jet mass, we ran the diff eq in $\tau$. This "resummed" the logarithms of $\tau$.

## For a parton shower, want a "universal" (observable-independent) measure

Exact choice is ambiguous. Dictates which specific "logs" our shower will resum.

No naked singularities: $Q$ must vanish in all unresolved (infrared and collinear) limits.

Reasonable to resum "biggest" (double) logs: motivates $Q^2 \sim \frac{1}{\text{dipole factor}} \sim \frac{s_{ij}s_{jk}}{s_{ijk}} \equiv p^2_{\perp A}$ "ARIADNE" $p_T$

Geometric mean of propagator virtualities

Used by VINCIA shower developed at Monash

(Note: other choices also possible, eg "angular ordering", other $p_T$ definitions, …)

# 3. Cast as iterative Markov-Chain algorithm

Standard Born-Level Matrix-Element calculation of $\mathrm{d}\sigma/d\mathcal{O}$ (for some generic observable $\mathcal{O}$):

Born $$\left.\frac{\mathrm{d}\sigma_H}{\mathrm{d}\mathcal{O}}\right|_{\mathrm{Born}} = \int \mathrm{d}\Phi_H \; |M_H^{(0)}|^2 \; \delta(\mathcal{O} - \mathcal{O}(\{p\}_H))$$

H = Hard process

{p} : partons

But instead of evaluating O directly on the Born final state, first insert a "showering operator"

Born + shower $$\left.\frac{\mathrm{d}\sigma_H}{\mathrm{d}\mathcal{O}}\right|_{\mathcal{S}} = \int \mathrm{d}\Phi_H \; |M_H^{(0)}|^2 \; \mathcal{S}(\{p\}_H, \mathcal{O})$$

{p} : partons

S : showering operator

Unitarity: to first order (in perturbation theory), S should do nothing:

$$\mathcal{S}(\{p\}_H, \mathcal{O}) = \delta\left(\mathcal{O} - \mathcal{O}(\{p\}_H)\right) \; + \; \mathcal{O}(\alpha_s)$$

**Actually, we know the all-orders probability that nothing happens:**

$$\Delta(t_1, t_2) = \exp\left( - \int_{t_1}^{t_2} \mathrm{d}t \, \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t} \right)$$

Sudakov Factor
(Exponentiation)

**Build this in, with $\mathrm{d}\Delta/dt$ = probability that state does change:**

(Markov Chain)

$$S(\{p\}_X, \mathcal{O}) = \Delta(t_{\mathrm{start}}, t_{\mathrm{had}}) \delta(\mathcal{O} - \mathcal{O}(\{p\}_X))$$

"Nothing Happens" → "Evaluate Observable"

$$- \int_{t_{\mathrm{start}}}^{t_{\mathrm{had}}} \mathrm{d}t \frac{\mathrm{d}\Delta(t_{\mathrm{start}}, t)}{\mathrm{d}t} S(\{p\}_{X+1}, \mathcal{O})$$

"Something Happens" → "Continue Shower"

# A Shower Algorithm*

*No time to explain Monte Carlo integration / sampling methods so must be taken on faith here

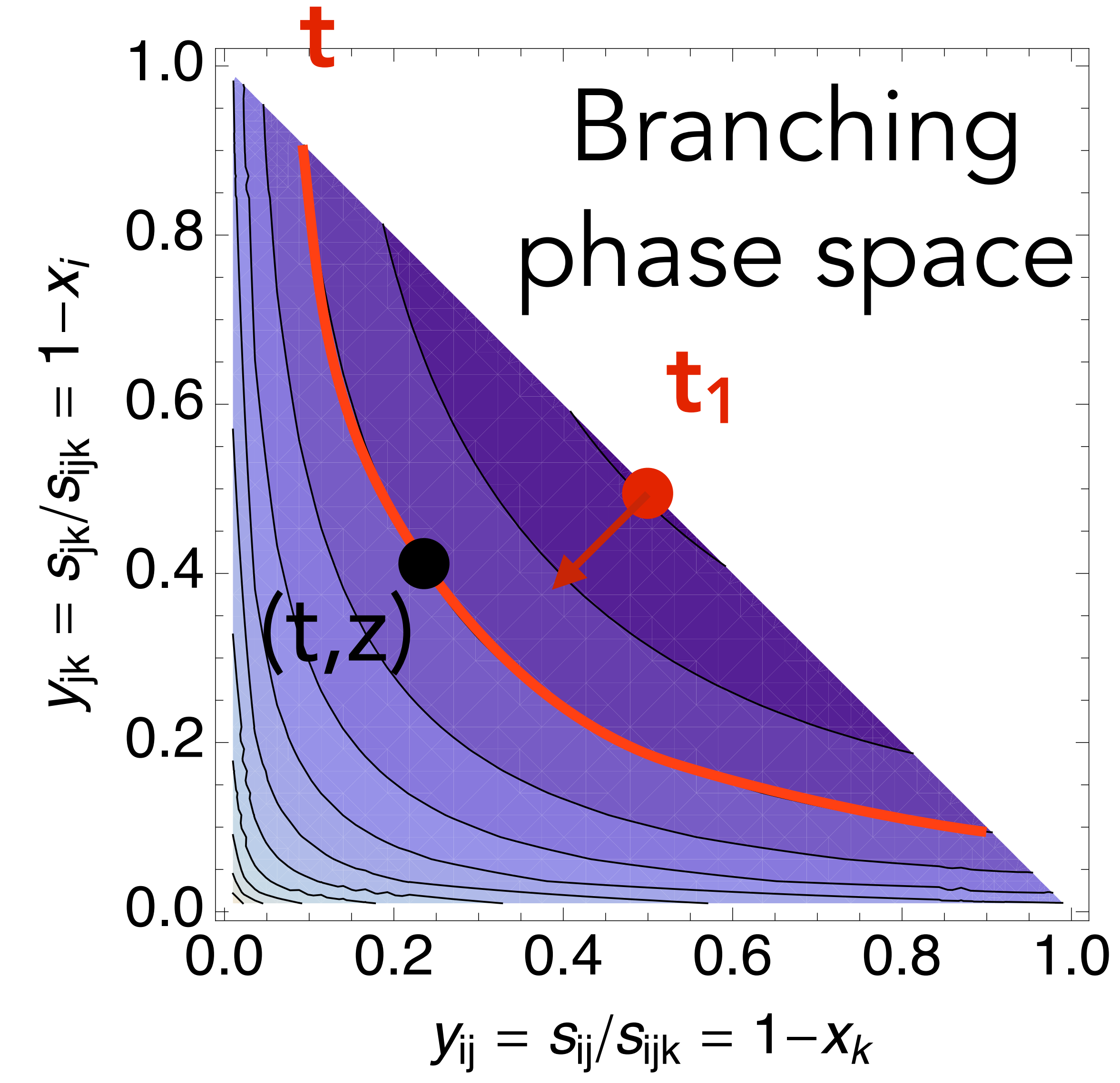
## 1. For each evolver, generate a random number $R \in [0,1]$

Solve equation $R = \Delta(t_1, t)$ for $t$ (with starting scale $t_1$)

Can be done analytically for simple splitting kernels,

else numerically and/or by trial + veto ("the veto algorithm")

→ stochastically sampled scale $t$ for next (trial) branching




## 2. Generate another Random Number, $R_z \in [0,1]$

To find second (linearly independent) phase-space invariant

Solve equation $R_z = \dfrac{I_z(z,t)}{I_z(z_{\max}(t),t)}$ for $z$ (at scale $t$), with $I_z(z,t) = \displaystyle\int_{z_{\min}(t)}^{z} \mathrm{d}z' \frac{\mathrm{d}\Delta(t')}{\mathrm{d}t}\bigg|_{t'=t}$

$I_z$ is called the "primitive function"

## 3. Generate a third Random Number, $R_\phi \in [0,1]$

Solve $R_\varphi = \varphi/(2\pi)$ for $\varphi$. Can now do 3D branching; construct tentative branched state.

Accept/Reject based on full kinematics. Update $t_1 = t$. Update state (if accept). **Repeat.**

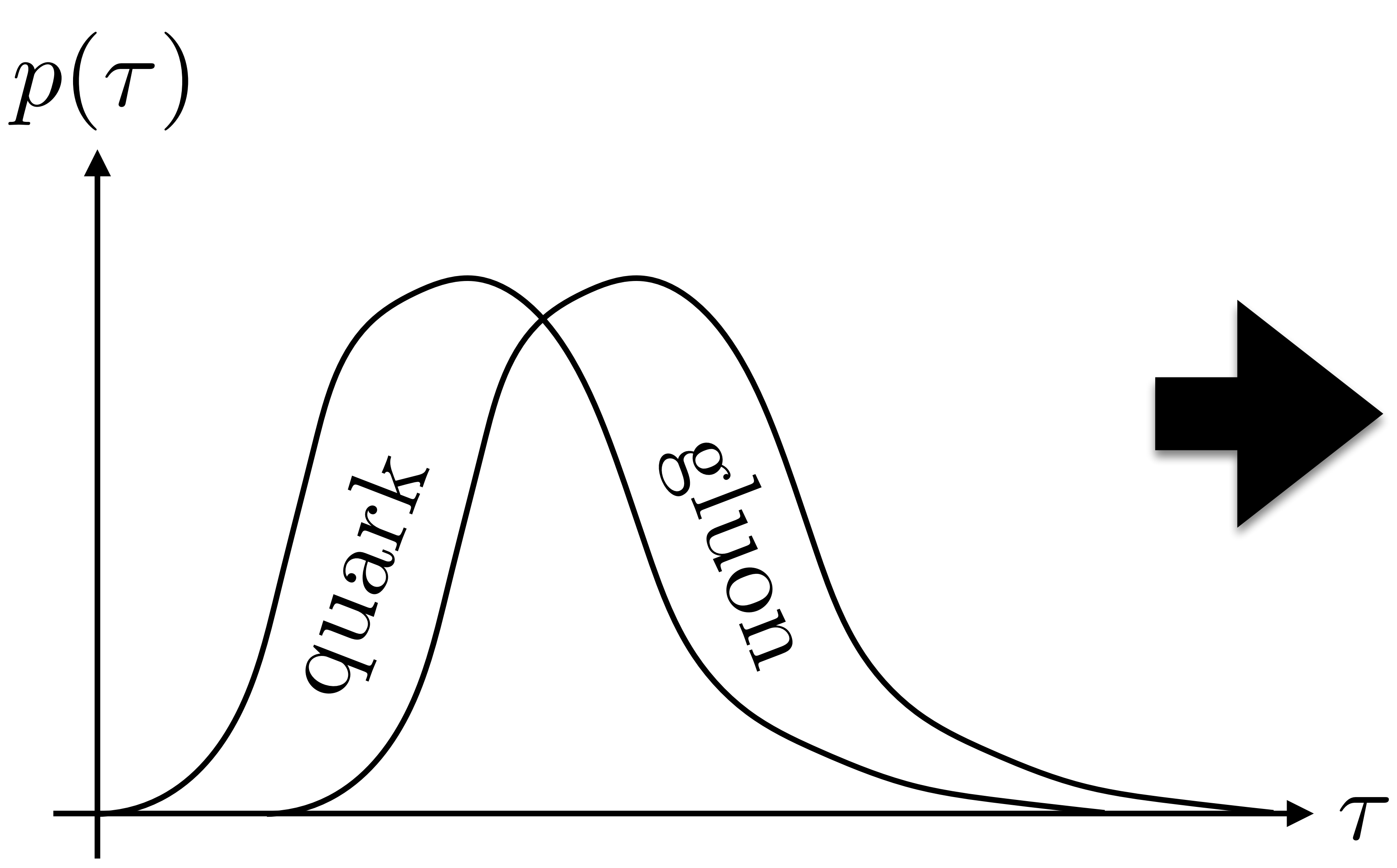
# Application: Quark-Gluon Jet Discrimination

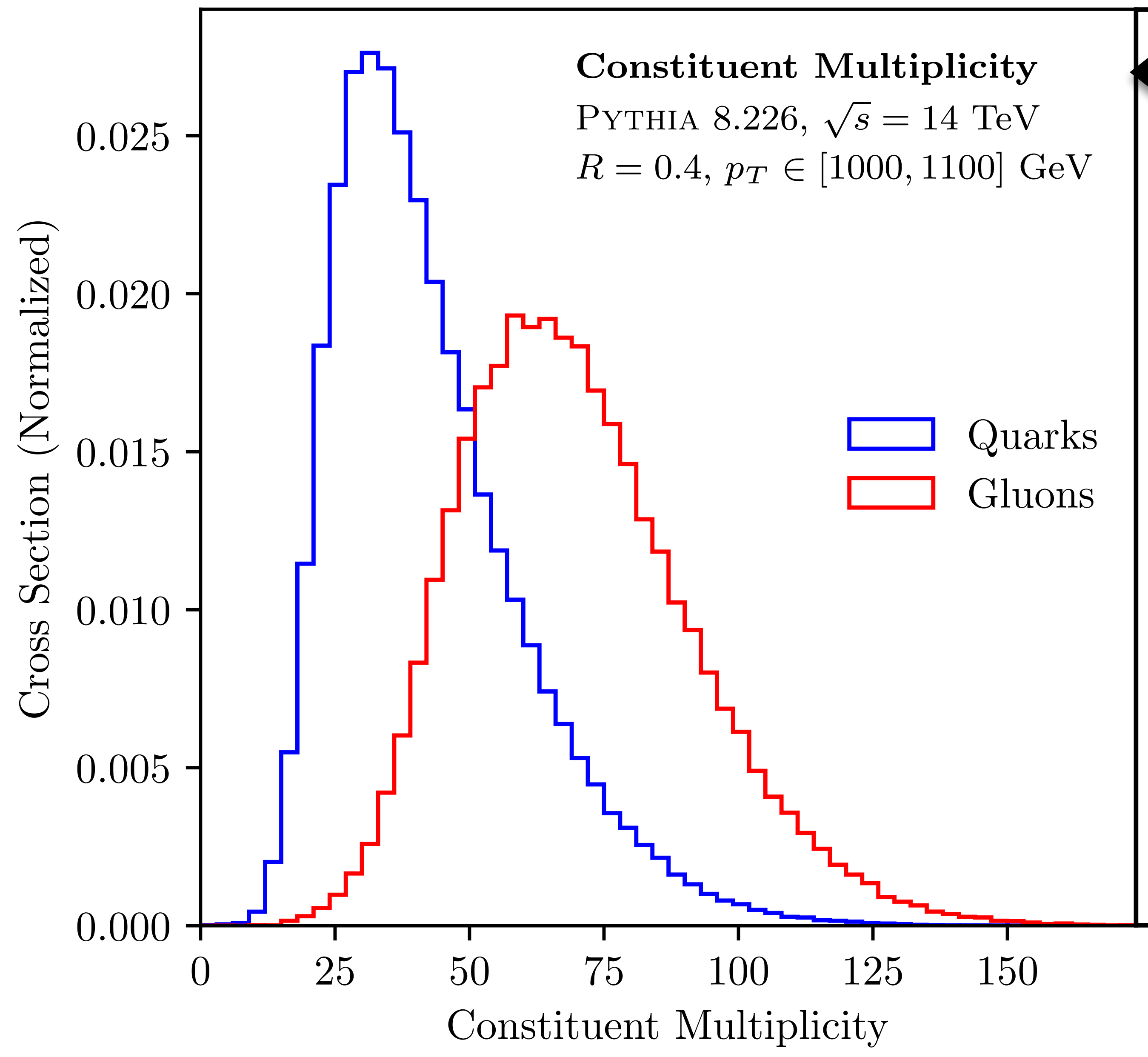
**Can use our simple jet-mass calculation to ask a fundamental question: can we tell a quark-initiated jet apart from a gluon-initiated one?**

Jet mass for quark-initiated jets: analytical result

$$p(\tau) = \frac{d}{d\tau} \exp\left[ -\frac{\alpha_s}{\pi}\frac{C_F}{2} \log^2 \tau \right] = -\frac{\alpha_s C_F}{\pi}\frac{\log \tau}{\tau} \exp\left[ -\frac{\alpha_s}{\pi}\frac{C_F}{2} \log^2 \tau \right] .$$

DLA → same result for gluon jets, but with octet colour charge Casimir $C_A \sim 2C_F$




Apologies; this is multiplicity not jet mass (did not have time to make new plot)

**(Parton-Shower Algorithms)**

Start from a Born-level parton configuration, or "hard process".

Simulate bremsstrahlung by stochastic sampling of Sudakov factors; adding branchings iteratively, ordered in decreasing "resolution" (e.g. ∝ $p_T$)

Can include full phase space, recoils, mass effects, running $\alpha_s$, …, matching to hadronisation models and even detector simulations.

**The workhorses of collider phenomenology**

# Extra Slides

## Probability Conservation: P(something happens) + P(nothing happens) = 1

### In Showers: Imposed by Event *evolution*: "detailed balance"

When (X) branches to (X+1): **Gain** one (X+1). **Lose** one (X). ➜ A "gain-loss" differential equation.

Cast as **iterative** (Markov-Chain Monte-Carlo) evolution algorithm, based on universality and unitarity.

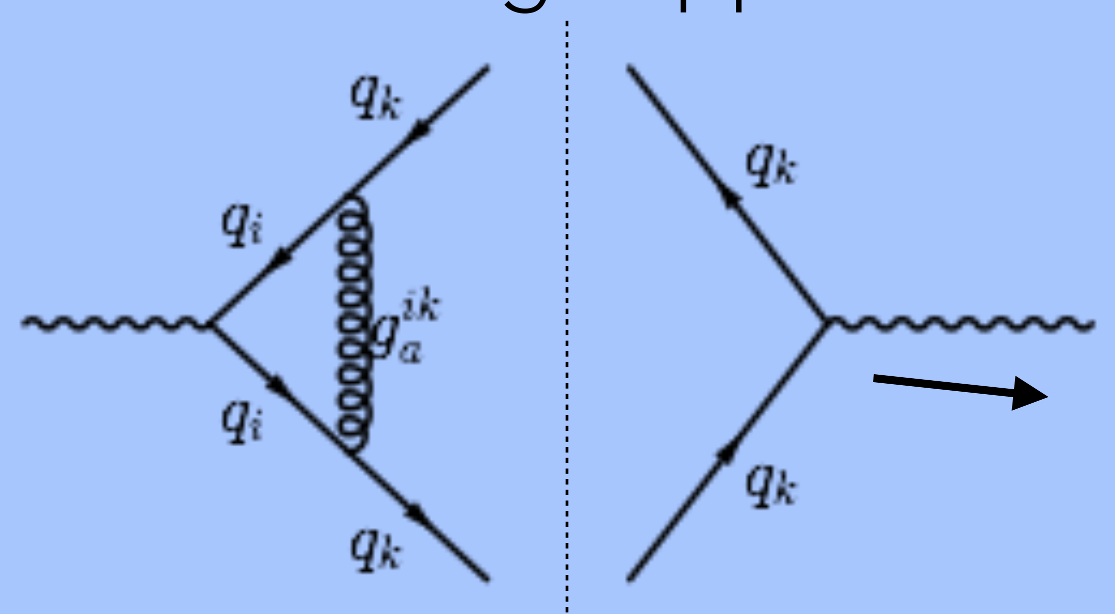
With evolution kernel ~ $\frac{|M_{n+1}|^2}{|M_n|^2}$ (typically a soft/collinear approx thereof)

Evolve in some measure of **resolution** ~ hardness, 1/time … ~ **fractal scale**

Typical choices $p_\perp, Q^2, E\theta, \ldots$

### Compare with NLO (e.g., in previous lecture)

"Nothing happens"

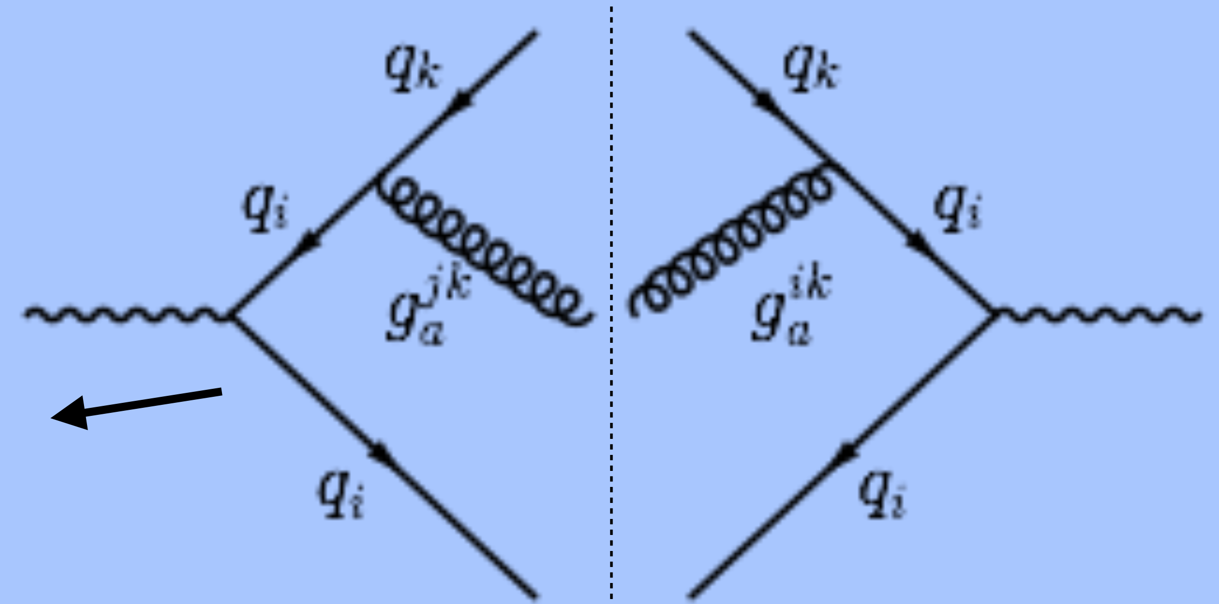



**KLN:** sum over degenerate quantum states = finite; infinities must cancel)

$$\text{Loop} = -\int \text{Tree} + F$$

*F* for "finite"

"something happens"




$2\text{Re}\left[\mathcal{M}^{(1)}\mathcal{M}^{(0)*}\right]$ *Showers neglect F → "Leading-Logarithmic" **(LL)** Approximation* $\left|\mathcal{M}^{(0)}_{+1}\right|^2$

# Optional: Gluons on the Lund Plane ➤ Origami Diagrams

Illustrations from Dreyer, Salam, Soyez, arXiv:1807.04758

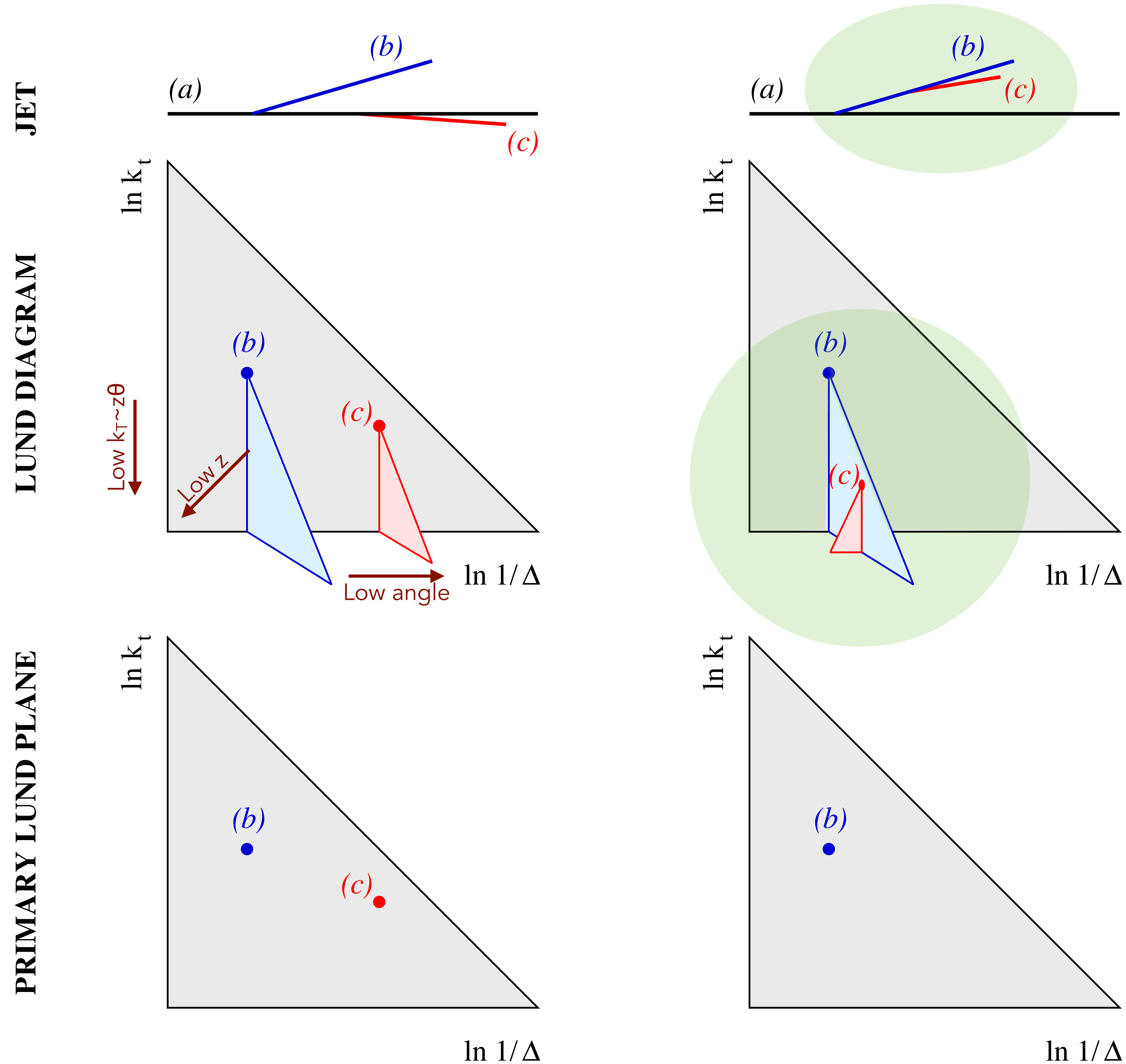
## Illustration

In QCD, gluons are themselves charged, so **can radiate further gluons**

➤ Each gluon "adds" new phase space

➤ Lund plane turns into an **"origami diagram"**

(Also note the vertical axis now goes the way)



# Optional: Measurement of the Lund Plane for QCD Jets

**Measurement of the Lund Jet Plane Using Charged Particles in 13 TeV Proton-Proton Collisions with the ATLAS Detector**

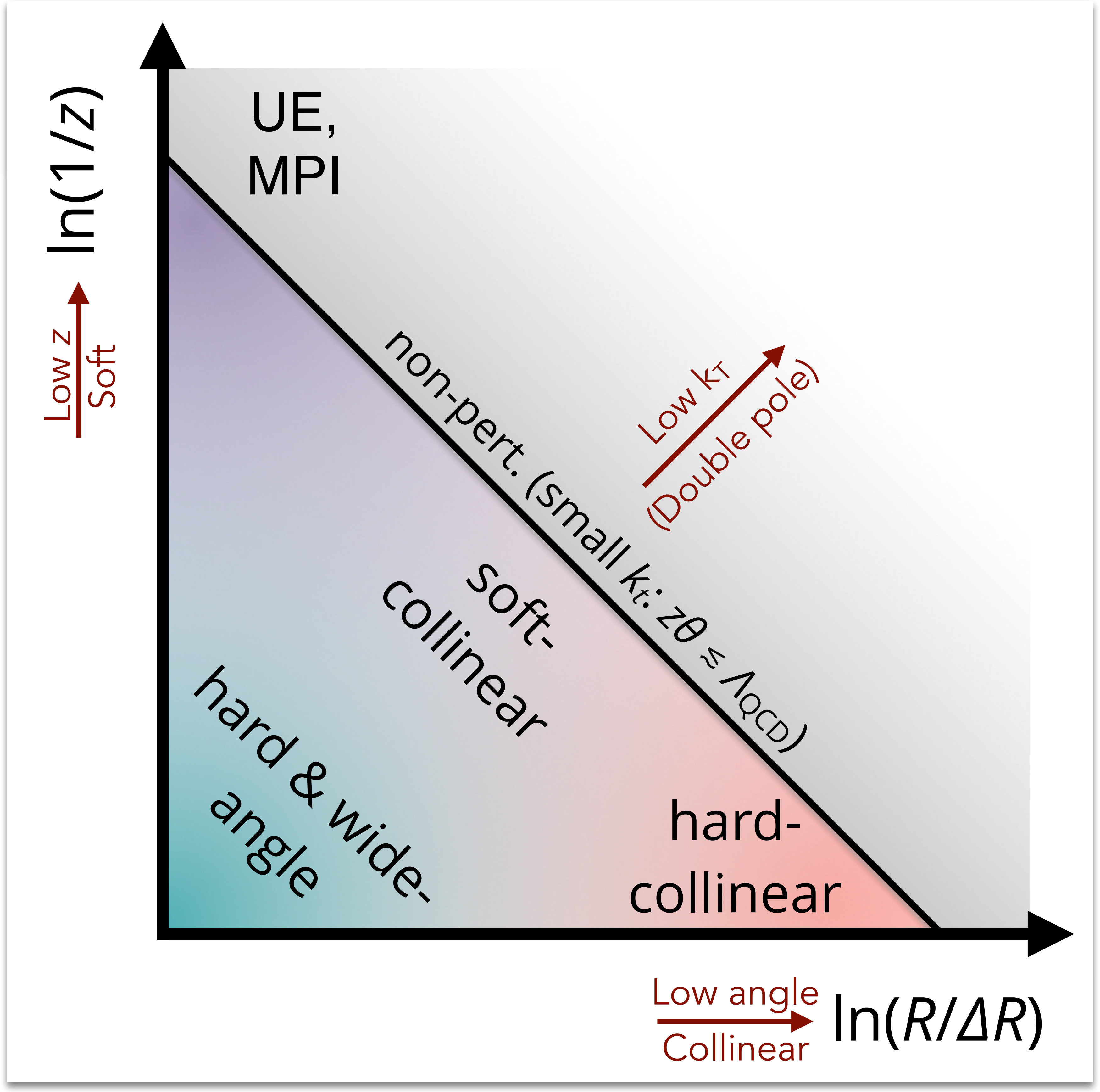
ATLAS Collaboration • Georges Aad (Marseille, CPPM) et al. (Apr 7, 2020)

Published in: *Phys.Rev.Lett.* 124 (2020) 22, 222002 • e-Print: 2004.03540 [hep-ex]

pdf   DOI   cite   datasets

Uniform density not *that* easy to see, in practice…

**Question(s): why?**




(a) Schematic representation of the LJP.