QFT Beyond Fixed Order

Introduction to Bremsstrahlung and Jets

1. Radiation from Accelerated Charges

Soft Bremsstrahlung in Classical E&M, and in QED. ***The dipole factor*** *& coherence.*

➜ 2. Infrared Singularities and Infrared Safety

IR Poles & Sudakov Logarithms. ***Probabilities > 1****.*

Summing over degenerate quantum states (KLN theorem). ***IRC Safety.***

3. QCD as a Weakly Coupled Conformal Field Theory

The ***emission*** *probability; Double-Logarithmic Approximation*

The ***no-emission*** *probability; Sudakov Factor; exponentiation; example:* ***jet mass****.*

4. Parton Showers

DLA as differential evolution kernels; unitarity and detailed balance.

Sampling the Sudakov; perturbation theory as a Markov Chain; Monte Carlo.

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The total probability to emit a photon?

Having done the φ integral, the total probability is now given by:

$$P_\gamma = \frac{e^2}{4\pi^2}\int_0^{k_{\max}} \frac{\mathrm{d}k}{\boxed{k}} \int_{\cos\theta_{pp'}}^{1} \frac{\mathrm{d}\cos\theta_k}{\boxed{(1-\cos\theta_k)}} \to \infty$$

We can artificially regulate this by introducing a $k_{\min}$ (formally equivalent to a photon mass) and a $\theta_{\min}$ (~ equivalent to a mass for the radiating particle)

$$\to \frac{e^2}{4\pi^2} \ln\left(\frac{k_{\max}}{k_{\min}}\right) \ln\left(\frac{1-\cos\theta_{pp'}}{1-\cos\theta_{\min}}\right)$$

The "Sudakov double logarithm"

Logarithmically divergent.

These are the (leading) **infrared divergences** of QED (also exist in QCD)

Interpretation as a probability has a problem

For sufficiently small $k_{\min}$ and/or $\theta_{\min}$, this probability becomes > 1

Interpreting Probabilities > 1

The Born-level $\sigma_{pp'}$ $\propto$ event rate for $p \to p'$ scattering

The radiative cross section $\sigma_{pp'+\gamma}$ $\propto$ photon rate in $p \to p'$ scattering

What would an experimentalist conclude if their photon detector was triggering at a higher rate than their *p'* detector?

Simply that each p' was accompanied by **more than one photon** on average!

$$\frac{\sigma_{pp'+\gamma}}{\sigma_{pp'}} = \langle n_\gamma \rangle_{pp'}$$

The regulator variables $k_{\min}$, $\theta_{\min}$ then represent an (arbitrary) definition of the smallest photon energies and angles we can **resolve** in a given context.

➤ **Expect** $\sigma_{pp'+\gamma}(k_{\min}, \theta_{\min})/\sigma_{pp'} \sim$ **number of "resolved" photons**

So … the total probability to emit a photon?

Still, if we want the total correction to the Born, we must include $k_{\rm min} \to 0,\ \theta_{\rm min} \to 0$

Looks like **total** probability to emit a photon (“resolved” or not) is infinite.

(Related to the infinite range of the Coulomb field $\leftrightarrow$ massless photon.)

But then … what about our QED perturbation expansion?

In perturbation theory, each higher-order term is supposed to be **smaller** than the previous one.

But it looks like our first-order QED correction is not only larger than the Born, it is **infinite**!

Perhaps not surprising given that bremsstrahlung is essentially a classical process; *ought* to involve an infinite number of quanta (correspondence).

Resolved and Unresolved Quantum States

We deal with UV divergences through renormalisation

Redefine couplings & fields to absorb anything smaller than wavelength of our probe

The analogy for **IR divergences is:**

"bare electron" and "bare photon" ➤ "electron + unresolved photons" and "resolved photons".

To the rescue:

1. In QM, we must **sum over degenerate quantum states**.

   Saves fixed-order perturbation theory (next slide).

2. Reinterpret divergent cross section for one emission in terms of divergent **number** of emissions, with finite total cross section

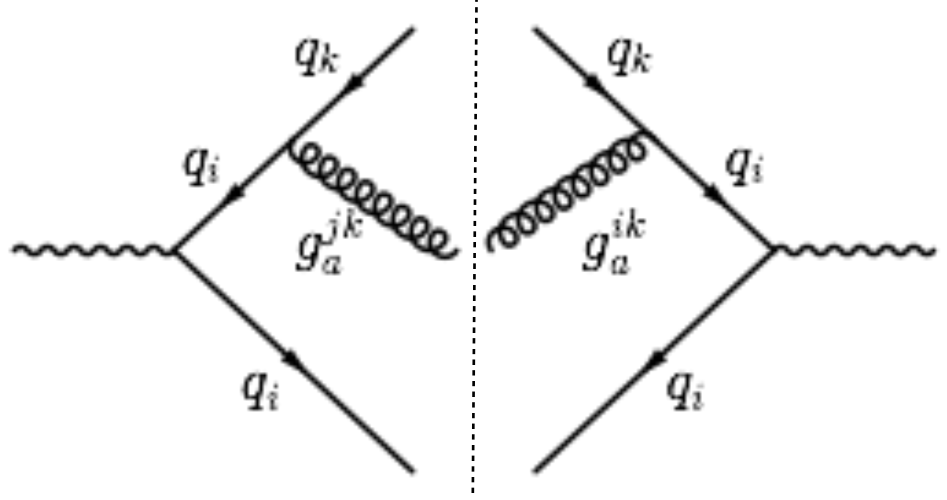
   ➤ **infinite-order** resummations & parton showers (next lectures).

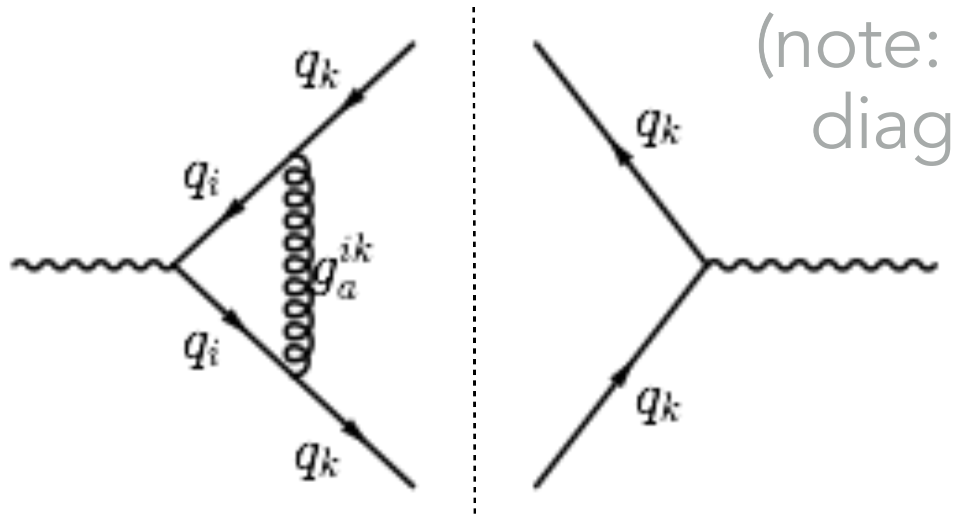
Sum over 'degenerate quantum states'

In the IR limit (Born + *infinitely* soft/collinear photon), the Born + γ final state is indistinguishable from the Born state.

Complete calculation must include both **emission** and **reabsorption** amplitudes

At NLO: $$\sigma_X^{\mathrm{NLO}} = \int |M_X^{(0)}|^2 + \int |M_{X+1}^{(0)}|^2 + \int 2\mathrm{Re}[M_X^{(1)} M_X^{(0)*}]$$





(note: **not** the 1-loop diagram squared)

Same IR singularities (from poles of propagators going on shell) but **opposite signs**!

(General proof beyond scope of this course)

➜ *Kinoshita-Lee-Nauenberg Theorem*: IR singularities cancel each other out, order by order:

E.g.: $$\sigma_{\mathrm{NLO}}(e^+e^- \to q\bar{q}) = \sigma_{\mathrm{LO}}(e^+e^- \to q\bar{q}) \left(1 + \frac{\alpha_s(E_{\mathrm{CM}})}{\pi} + \mathcal{O}(\alpha_s^2)\right)$$

Note: the equation on the previous slide was written quite schematically:

At NLO: $\sigma_{\mathrm{X}}^{\mathrm{NLO}} = \int |M_X^{(0)}|^2 + \int |M_{X+1}^{(0)}|^2 + \int 2\mathrm{Re}[M_X^{(1)} M_X^{(0)*}]$

Really, $\sigma_X^{\mathrm{NLO}} = \int \left| M_X^{(0)} \right|^2 \mathrm{d}\Phi_X + \int \left| M_{X+1}^{(0)} \right|^2 \mathrm{d}\Phi_{X+1} + \int 2\mathrm{Re}\left[M_X^{(1)} M_X^{(0)*} \right] \mathrm{d}\Phi_X$

Lorentz-Invariant Phase Spaces

Note: should really also show flux factor, symmetry/averaging factors, and PDF factors

Can also write: $\sigma_X^{\mathrm{NLO}} = \int \mathrm{d}\sigma_X^{\mathrm{B}} + \int \mathrm{d}\sigma_{X+1}^{\mathrm{R}} + \int \mathrm{d}\sigma_X^{\mathrm{V}}$

"Born" (LO) — "Real" (NLO) — "Virtual" (NLO)

(The Subtraction Approach)

Add and subtract universal IR limits (SOFT and COLLINEAR)

$$\mathrm{d}\sigma_{NLO} = \int_{\mathrm{d}\Phi_{m+1}} \underbrace{\big(\overset{|M^{(0)}_{X+1}|^2}{\mathrm{d}\sigma^R_{NLO}} - \boxed{\mathrm{d}\sigma^S_{NLO}}\big)}_{\text{Finite by Universality}} + \underbrace{\left[\int_{\mathrm{d}\Phi_{m+1}} \boxed{\mathrm{d}\sigma^S_{NLO}} + \int_{\mathrm{d}\Phi_m} \overset{2\mathrm{Re}[M^{(0)}_X M^{(1)*}_X]}{\mathrm{d}\sigma^V_{NLO}}\right]}_{\text{Finite by KLN}}$$

Finite by KLN
Compute and tabulate (regulated) $\mathrm{d}\sigma^S_{\mathrm{NLO}}$ integrals once and for all

Dipoles (Catani-Seymour)
Global Antennae (Gehrmann, Gehrmann-de Ridder, Glover)
Sector Antennae (Kosower, Peskin&Larkoski, …)
…

Choice of "subtraction terms" $\mathrm{d}\sigma^{\mathrm{S}}_{\mathrm{NLO}}$:

Singularities mandated by gauge theory

Non-singular terms: up to you (added and subtracted here, so zero net contribution)

$$\frac{|\mathcal{M}(Z^0 \to q_i g_j \bar{q}_k)|^2}{|\mathcal{M}(Z^0 \to q_I \bar{q}_K)|^2} = g_s^2\, 2C_F \left[\overbrace{\frac{2 s_{ik}}{s_{ij} s_{jk}}}^{\text{SOFT}} + \overbrace{\frac{1}{s_{IK}} \left(\frac{s_{ij}}{s_{jk}} + \frac{s_{jk}}{s_{ij}} \right)}^{\text{COLLINEAR}} \right]$$

$$\frac{|\mathcal{M}(H^0 \to q_i g_j \bar{q}_k)|^2}{|\mathcal{M}(H^0 \to q_I \bar{q}_K)|^2} = g_s^2\, 2C_F \left[\underbrace{\frac{2 s_{ik}}{s_{ij} s_{jk}}}_{\text{SOFT}} + \underbrace{\frac{1}{s_{IK}} \left(\frac{s_{ij}}{s_{jk}} + \frac{s_{jk}}{s_{ij}} \right.}_{\text{COLLINEAR}} \underbrace{\left. + 2 \right)}_{\text{+F}} \right]$$

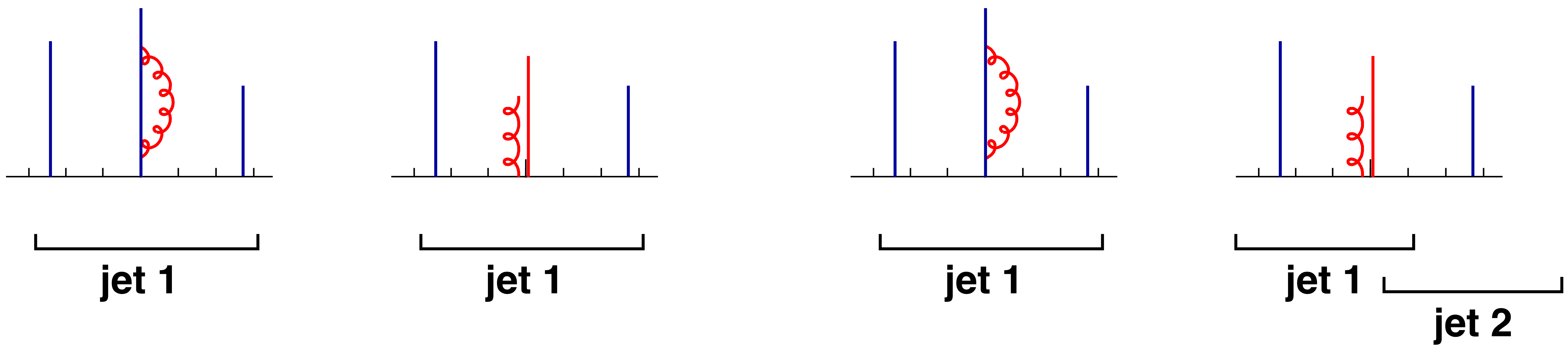
Not all observables can be computed perturbatively

All modern collider experiments use "infrared and collinear safe" jet clustering algorithms

But this was not always so. E.g., seeded cone algorithms (used at Tevatron) were not collinear safe.

(example by G. Salam)

Collinear Safe	**Collinear Unsafe**
Virtual and Real go into **same bins!**	Virtual and Real go into **different bins!**



$\alpha_s^n \times (-\infty)$ $\quad$ $\alpha_s^n \times (+\infty)$ $\quad$ $\alpha_s^n \times (-\infty)$ $\quad$ $\alpha_s^n \times (+\infty)$

Infinities cancel	**Infinities do not cancel**
(KLN: 'degenerate states')	Invalidates perturbation theory

(Note: in real life, **hadronisation scale** regulates the perturbative divergences in QCD.

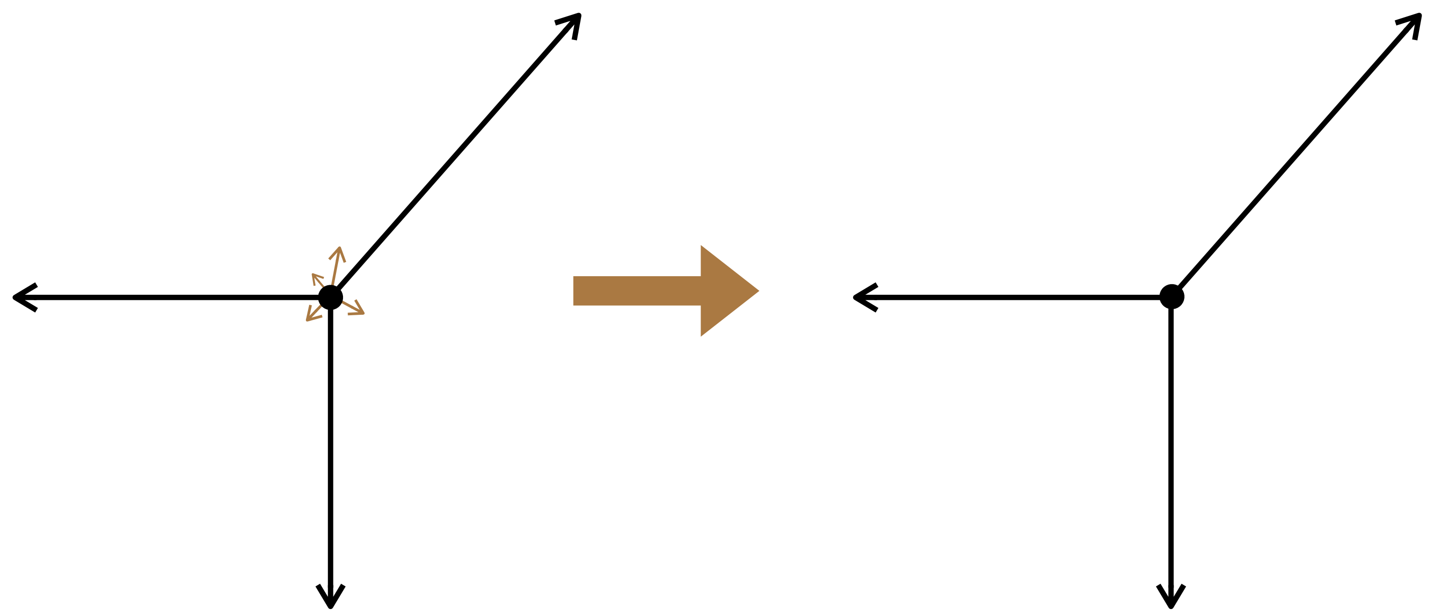
$\Longrightarrow$ What this means in practice is that **IRC safe** observables are relatively **insensitive** to hadronisation effects (they are suppressed by powers of Λ/Q), whereas **IRC unsafe** ones are **sensitive** to hadronisation effects.)

Perturbatively Calculable $\Longleftrightarrow$ "Infrared Safe"

Definition: An observable is **infrared safe** if it is **insensitive** to

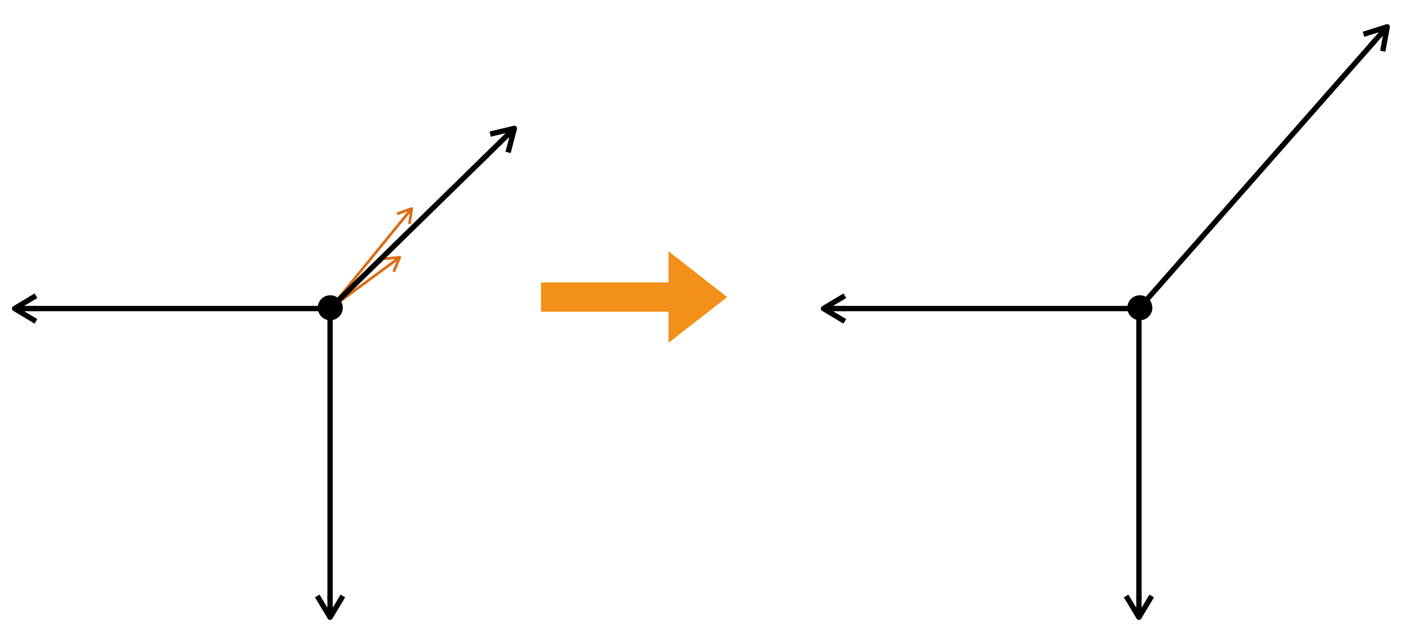
SOFT radiation:

Adding one (or more) infinitely ***soft* particles** (zero-energy) should not change the value of the observable



COLLINEAR splittings:

Splitting an existing particle up into **two** (or more) ***comoving* ones** (conserving total momentum and energy) should not change the value of the observable



→ ensures that virtual and real singularities go in "same bin" (of histograms), and hence cancel

→ observable can be computed perturbatively & hadronisation effects suppressed by $(\Lambda/Q)^n$

Note on terminology

My usage of the terms infrared, soft, and collinear:

$$\text{Infrared} \begin{cases} \text{Soft} \\ \text{Collinear} \end{cases}$$

i.e., Infrared = Soft and/or Collinear

Consistent with general distinction between UV and IR singularities in QFT.

Thus, if I say an observable is "IR safe", it is both soft and collinear safe.

Most others follow a historical convention:

Infrared only means soft

To cover both cases one then has to say "Infrared and Collinear Safe".

Gets abbreviated to **IRC Safe** which is what you'll often see in literature.

IRC Safety: Examples

Discuss whether the following observables are both soft and collinear safe, or not.

For those questions that involve *jets*, assume an arbitrary but IRC safe jet definition.

A) The **number** of **particles** (in an event).

B) The **number** of **jets** (in an event).

C) The **energy** of the **hardest particle** (in an event).

D) The $\mathbf{p_T}$ of the **hardest particle** …

E) The $\mathbf{p_T}$ of the **hardest jet** …

F) The **number** of **particles** with energies $E \geq E_{min}$, for some given E_{min}

G) The **summed** $\mathbf{p_T}$ of **all jets** (also called H_T)

Extra Slides