Introduction to QCD

1. Fundamentals of QCD
2. PDFs, Fixed-Order QCD, and Jet Algorithms
3. Parton Showers and Event Generators
4. QCD in the Infrared

Slides posted at:
www.cern.ch/skands/slides

Lecture Notes:
P. Skands, arXiv:1207.2389
Calculate Everything $\approx$ solve QCD $\rightarrow$ requires compromise!

Improve lowest-order perturbation theory, by including the ‘most significant’ corrections $\rightarrow$ complete events (can evaluate any observable you want)

The Workhorses

+ MORE SPECIALIZED: ALPGEN, MADGRAPH, HELAC, ARIADNE, VINCIA, WHIZARD, (a)MC@NLO, POWHEG, HEJ, PHOJET, EPOS, QGSJET, SIBYLL, DPMJET, LDCMC, DIPSY, HIJING, CASCADE, GOSAM, BLACKHAT, …

Reality is more complicated
PYTHIA anno 1978
(then called JETSET)

A Monte Carlo Program for Quark Jet Generation

T. Sjöstrand, B. Söderberg

A Monte Carlo computer program is presented, that simulates the fragmentation of a fast parton into a jet of mesons. It uses an iterative scaling scheme and is compatible with the jet model of Field and Feynman.

Note:
Field-Feynman was an early fragmentation model
Now superseded by the String (in PYTHIA) and Cluster (in HERWIG & SHERPA) models.
PYTHIA anno 2013
(now called PYTHIA 8)

The Pythia program is a standard tool for the generation of high-energy collisions, comprising a coherent set of physics models for the evolution from a few-body hard process to a complex multihadronic final state. It contains a library of hard processes and models for initial- and final-state parton showers, multiple parton-parton interactions, beam remnants, string fragmentation and particle decays. It also has a set of utilities and interfaces to external programs.
Divide and Conquer

Factorization → Split the problem into many (nested) pieces
+ Quantum mechanics → Probabilities → Random Numbers

\[ P_{\text{event}} = P_{\text{hard}} \otimes P_{\text{dec}} \otimes P_{\text{ISR}} \otimes P_{\text{FSR}} \otimes P_{\text{MPI}} \otimes P_{\text{Had}} \otimes \ldots \]

**Hard Process & Decays:**
- Use (N)LO matrix elements
- Sets “hard” resolution scale for process: \( Q_{\text{MAX}} \)

**Initial- & Final-State Radiation (ISR & FSR):**
- Altarelli-Parisi equations → differential evolution, \( dP/dQ^2 \), as function of resolution scale; run from \( Q_{\text{MAX}} \) to \( \sim 1 \) GeV (This Lecture)

**MPI (Multi-Parton Interactions)**
- Additional (soft) parton-parton interactions: LO matrix elements
- Additional (soft) “Underlying-Event” activity

**Hadronization**
- Non-perturbative model of color-singlet parton systems → hadrons
Recall: Jets ≈ Fractals

- Most bremsstrahlung is driven by divergent propagators → simple structure
- Amplitudes factorize in singular limits (→ universal "conformal" or "fractal" structure)

Partons $ab$ → "collinear":

$$|\mathcal{M}_{F+1}(\ldots, a, b, \ldots)|^2 \xrightarrow{a||b} g_s^2 C \frac{P(z)}{2(p_a \cdot p_b)} |\mathcal{M}_F(\ldots, a + b, \ldots)|^2$$

Gluon $j$ → "soft":

$$|\mathcal{M}_{F+1}(\ldots, i, j, k, \ldots)|^2 \xrightarrow{j \to 0} g_s^2 C \frac{(p_i \cdot p_k)}{(p_i \cdot p_j)(p_j \cdot p_k)} |\mathcal{M}_F(\ldots, i, k, \ldots)|^2$$

+ scaling violation: $g_s^2 \to 4\pi\alpha_s(Q^2)$

$P(z) =$ DGLAP splitting kernels, with $z =$ energy fraction $= E_a/(E_a+E_b)$

Coherence → Parton $j$ really emitted by $(i,k)$ "colour antenna"

Can apply this many times → nested factorizations
Bremsstrahlung

For any basic process $d\sigma_X = \sqrt{ }$ (calculated process by process)

$$d\sigma_{X+1} \sim N C g_s^2 \frac{d s_{i1}}{s_{i1}} \frac{d s_{1j}}{s_{1j}} d\sigma_X \sqrt{ }$$

$$d\sigma_{X+2} \sim N C g_s^2 \frac{d s_{i2}}{s_{i2}} \frac{d s_{2j}}{s_{2j}} d\sigma_{X+1} \sqrt{ }$$

$$d\sigma_{X+3} \sim N C g_s^2 \frac{d s_{i3}}{s_{i3}} \frac{d s_{3j}}{s_{3j}} d\sigma_{X+2} \ldots$$

Factorization in Soft and Collinear Limits

\[ P(z) : \text{“DGLAP Splitting Functions”} \]

\[ |M(\ldots, p_i, p_j \ldots)|^2 \xrightarrow{\parallel j} g_s^2 C \frac{P(z)}{s_{ij}} |M(\ldots, p_i + p_j, \ldots)|^2 \]

\[ |M(\ldots, p_i, p_j, p_k \ldots)|^2 \xrightarrow{jg \rightarrow 0} g_s^2 C \frac{2s_{ik}}{s_{ij}s_{jk}} |M(\ldots, p_i, p_k, \ldots)|^2 \]

“Soft Eikonal” : generalizes to Dipole/Antenna Functions (more later)
Bremsstrahlung

For any basic process \( d\sigma_X = \checkmark \) (calculated process by process)

\[
d\sigma_{X+1} \sim N_C 2g_s^2 \frac{ds_{i1}}{s_{i1}} \frac{ds_{1j}}{s_{1j}} d\sigma_X \checkmark \\
d\sigma_{X+2} \sim N_C 2g_s^2 \frac{ds_{i2}}{s_{i2}} \frac{ds_{2j}}{s_{2j}} d\sigma_{X+1} \checkmark \\
d\sigma_{X+3} \sim N_C 2g_s^2 \frac{ds_{i3}}{s_{i3}} \frac{ds_{3j}}{s_{3j}} d\sigma_{X+2} \checkmark ...
\]

Singularities: mandated by gauge theory
Non-singular terms: process-dependent

\[
\frac{|M(Z^0 \rightarrow q_i g_j \bar{q}_k)|^2}{|M(Z^0 \rightarrow q_I \bar{q}_K)|^2} = g_s^2 2C_F \quad \text{SOFT} \quad \frac{2s_{ik}}{s_{ij}s_{jk}} + \frac{1}{s_{IK}} \left( \frac{s_{ij}}{s_{jk}} + \frac{s_{jk}}{s_{ij}} \right)
\]

\[
\frac{|M(H^0 \rightarrow q_i g_j \bar{q}_k)|^2}{|M(H^0 \rightarrow q_I \bar{q}_K)|^2} = g_s^2 2C_F \quad \text{SOFT} \quad \frac{2s_{ik}}{s_{ij}s_{jk}} + \frac{1}{s_{IK}} \left( \frac{s_{ij}}{s_{jk}} + \frac{s_{jk}}{s_{ij}} + 2 \right)
\]
For any basic process $d\sigma_X = \checkmark$ (calculated process by process)

$$d\sigma_{X+1} \sim N_C 2 g_s^2 \frac{ds_{i1}}{s_{i1}} \frac{ds_{1j}}{s_{1j}} d\sigma_X \checkmark$$

$$d\sigma_{X+2} \sim N_C 2 g_s^2 \frac{ds_{i2}}{s_{i2}} \frac{ds_{2j}}{s_{2j}} d\sigma_{X+1} \checkmark$$

$$d\sigma_{X+3} \sim N_C 2 g_s^2 \frac{ds_{i3}}{s_{i3}} \frac{ds_{3j}}{s_{3j}} d\sigma_{X+2} \ldots$$

**Iterated factorization**

Gives us a universal approximation to $\infty$-order tree-level cross sections.

Exact in singular (strongly ordered) limit.

Finite terms (non-universal) $\rightarrow$ Uncertainties for non-singular (hard) radiation

But something is not right … Total $\sigma$ would be infinite …
Loops and Legs

Coefficients of the Perturbative Series

Loops

X  X+1  ...  

X  X+1  X+2  X+3  ...

Legs

Born  X+1  X+2  X+3  ...

The corrections from Quantum Loops are missing

Universality (scaling)

Jet-within-a-jet-within-a-jet-...
Unitarity: \( \text{sum(probability)} = 1 \)

Kinoshita-Lee-Nauenberg (Lecture 2):
(sum over degenerate quantum states = finite)

\[
\text{Loop} = -\, \text{Int(Tree)} + F
\]

Parton Showers neglect \( F \) \( \rightarrow \) Leading-Logarithmic (LL) Approximation

**Imposed by Event evolution:**

When \( (X) \) branches to \( (X+1) \): **Gain** one \( (X+1) \). **Loose** one \( (X) \).

\[
\rightarrow \text{evolution equation with kernel} \quad \frac{d\sigma_{X+1}}{d\sigma_X}
\]

Evolve in some measure of resolution \( \sim \) hardness, 1/time … \( \sim \) fractal scale

\( \rightarrow \) includes both real (tree) and virtual (loop) corrections
Interpretation: the structure evolves! (example: $X = 2$-jets)

- Take a jet algorithm, with resolution measure “Q”, apply it to your events
- At a very crude resolution, you find that everything is 2-jets
What we need is a differential equation

Boundary condition: a few partons defined at a high scale ($Q_F$)
Then evolves (or “runs”) that parton system down to a low scale (the hadronization cutoff $\sim 1$ GeV) $\rightarrow$ It’s an evolution equation in $Q_F$

Close analogue: nuclear decay
Evolve an unstable nucleus. Check if it decays + follow chains of decays.

\[
\begin{align*}
\text{Decay constant} & \quad \frac{dP(t)}{dt} = c_N \\
\text{Probability to remain undecayed in the time interval} \ [t_1,t_2] & \quad \Delta(t_1,t_2) = \exp \left( - \int_{t_1}^{t_2} c_N \ dt \right) = \exp (-c_N \Delta t) \\
\text{Decay probability per unit time} & \quad \frac{dP_{\text{res}}(t)}{dt} = -\frac{d\Delta}{dt} = c_N \Delta(t_1,t) \\
(\text{requires that the nucleus did not already decay}) & \quad \Delta(t_1,t_2) : \text{“Sudakov Factor”}
\end{align*}
\]
Nuclear Decay

\[ S(\{p\} X, O) = \delta(O - O(\{p\} X)) \]

\[ S(\{p\} X, O) = \Delta(t_{\text{start}}, t_{\text{had}}) \delta(O - O(\{p\} X)) - \int_{t_{\text{had}}}^{t_{\text{start}}} d\tau \int dP \frac{dP}{dt} \]

\[ P = \int d\Phi X + 1 d\Phi X w X + 1 w X \]

\[ P_{\text{DGLAP}} = \sum_i \int dQ^2 Q^2 dz P_i(z) \]

\[ \Delta(t_1, t_2) = \exp \left( - \int_{t_1}^{t_2} dt \frac{dP}{dt} \right) \]

Nuclei remaining undecayed after time t

First Order

Second Order

Third Order

Exponential

Early Times

Time

Late Times

Nuclei remaining undecayed after time t

= \Delta(t_1, t_2) = \exp \left( - \int_{t_1}^{t_2} dt \frac{dP}{dt} \right)
The Sudakov Factor

In nuclear decay, the Sudakov factor counts:

How many nuclei remain undecayed after a time $t$

Probability to remain undecayed in the time interval $[t_1, t_2]$

$$\Delta(t_1, t_2) = \exp \left( - \int_{t_1}^{t_2} c_N \, dt \right) = \exp (-c_N \Delta t)$$

The Sudakov factor for a parton system counts:

The probability that the parton system doesn’t evolve (branch) when we run the factorization scale ($\sim 1/time$) from a high to a low scale

Evolution probability per unit “time”

$$\frac{dP_{\text{res}}(t)}{dt} = -\frac{d\Delta}{dt} = c_N \Delta(t_1, t)$$

(replace $t$ by shower evolution scale)

(replace $c_N$ by proper shower evolution kernels)
What’s the evolution kernel?

DGLAP splitting functions

Can be derived from *collinear limit* of MEs \((p_b+p_c)^2 \to 0\)

+ evolution equation from invariance with respect to \(Q_F \to \text{RGE}\)

\[
d\mathcal{P}_a = \sum_{b,c} \frac{\alpha_{abc}}{2\pi} P_{a \to bc}(z) \, dt \, dz .
\]

\[
P_{q\to qg}(z) = C_F \frac{1 + z^2}{1 - z} ,
\]

\[
P_{g\to gg}(z) = N_C \frac{(1 - z)(1 - z)}{z(1 - z)} ,
\]

\[
P_{g\to q\bar{q}}(z) = T_R \left( z^2 + (1 - z)^2 \right) ,
\]

\[
P_{q\to q\gamma}(z) = e_q^2 \frac{1 + z^2}{1 - z} ,
\]

\[
P_{\ell\to \ell\gamma}(z) = e_{\ell}^2 \frac{1 + z^2}{1 - z} ,
\]

... with \(Q^2\) some measure of “hardness”

= event/jet resolution

measuring parton virtualities / formation time / …

**Note:** there exist now also alternatives to AP kernels (with same collinear limits!): dipoles, antennae, ...

cf. conformal (fractal) QCD, Lecture 1
(and PDF evolution, Lecture 2)
Coherence

QED: Chudakov effect (mid-fifties)

\[
\text{cosmic ray } \gamma \text{ atom} \quad e^+ \quad e^-
\]

Approximations to Coherence:
- Angular Ordering (HERWIG)
- Angular Vetos (PYTHIA)
- Coherent Dipoles/Antennae (ARIADNE, Catani-Seymour, VINCIA)

QCD: colour coherence for soft gluon emission

\[
\begin{vmatrix}
\text{emulsion plate} & \text{reduced} & \text{normal}
\end{vmatrix}
\begin{vmatrix}
\text{ionization} & \text{ionization}
\end{vmatrix}
\]

→ an example of an interference effect that can be treated probabilistically

More interference effects can be included by matching to full matrix elements
Example: quark-quark scattering in hadron collisions

Consider one specific phase-space point (e.g., scattering at 45°)

2 possible colour flows: a and b

Figure 4: Angular distribution of the first gluon emission in $qq \rightarrow qq$ scattering at 45°, for the two different color flows. The light (red) histogram shows the emission density for the forward flow, and the dark (blue) histogram shows the emission density for the backward flow.

Another good recent example is the SM contribution to the Tevatron top-quark forward-backward asymmetry from coherent showers, see: PS, Webber, Winter, JHEP 1207 (2012) 151
**Initial-State vs Final-State Evolution**

**FSR:**
- Virtualities are Timelike: $p^2 > 0$
- Start at $Q^2 = Q_F^2$
- "Forwards evolution"

**ISR:**
- Virtualities are Spacelike: $p^2 < 0$
- $p^2 = t < 0$
- Start at $Q^2 = Q_F^2$
- Constrained backwards evolution towards boundary condition = proton

Separation meaningful for collinear radiation, but not for soft …
DGLAP for Parton Density

\[
\frac{df_b(x, t)}{dt} = \sum_{a,c} \int \frac{dx'}{x'} f_a(x', t) \frac{\alpha_{abc}}{2\pi} \frac{\alpha_{abc}(t')}{2\pi} P_{a\rightarrow bc} \left( \frac{x}{x'} \right)
\]

→ Sudakov for ISR

Contains a ratio of PDFs

\[
\Delta(x, t_{\text{max}}, t) = \exp \left\{ -\int_t^{t_{\text{max}}} dt' \sum_{a,c} \int \frac{dx'}{x'} \frac{f_a(x', t')}{f_b(x, t')} \frac{\alpha_{abc}(t')}{2\pi} P_{a\rightarrow bc} \left( \frac{x}{x'} \right) \right\}
\]

\[
= \exp \left\{ -\int_t^{t_{\text{max}}} dt' \sum_{a,c} \int dz \frac{\alpha_{abc}(t')}{2\pi} P_{a\rightarrow bc} \left( z \right) \frac{x' f_a(x', t')}{x f_b(x, t')} \right\}
\]
A tricky aspect for many parton showers. Illustrates that quantum ≠ classical!

**Who emitted that gluon?**

Real QFT = sum over amplitudes, then square → interference (IF coherence)

Respected by dipole/antenna languages (and by angular ordering), but not by conventional DGLAP (→ all PDFs are “wrong”)

Separation meaningful for collinear radiation, but not for soft …
Start from an **arbitrary lowest-order** process (green = QFT amplitude squared)

**Parton showers** generate the (LL) bremsstrahlung terms of the rest of the perturbative series (approximate infinite-order resummation)

**Note! LL ≠ full QCD! (→ matching)**
The Shower Operator

\[
\text{Born} \quad \left. \frac{d\sigma_H}{d\mathcal{O}} \right|_{\text{Born}} = \int d\Phi_H \ |M_H^{(0)}|^2 \delta(\mathcal{O} - \mathcal{O}(\{p\}_H)) \quad \{p\} : \text{partons}
\]

But instead of evaluating \(\mathcal{O}\) directly on the Born final state, first insert a showering operator

\[
\text{Born + shower} \quad \left. \frac{d\sigma_H}{d\mathcal{O}} \right|_{\mathcal{S}} = \int d\Phi_H \ |M_H^{(0)}|^2 \mathcal{S}(\{p\}_H, \mathcal{O}) \quad \{p\} : \text{partons} \quad \mathcal{S} : \text{showering operator}
\]

Unitarity: to first order, \(\mathcal{S}\) does nothing

\[
\mathcal{S}(\{p\}_H, \mathcal{O}) = \delta(\mathcal{O} - \mathcal{O}(\{p\}_H)) + \mathcal{O}(\alpha_s)
\]
The Shower Operator

To ALL Orders

\[ S(\{p\}X, \mathcal{O}) = \Delta(t_{\text{start}}, t_{\text{had}})\delta(\mathcal{O} - \mathcal{O}(\{p\}X)) \]

“Nothing Happens” \rightarrow “Evaluate Observable”

\[
- \int_{t_{\text{start}}}^{t_{\text{had}}} dt \frac{d\Delta(t_{\text{start}}, t)}{dt} S(\{p\}X+1, \mathcal{O})
\]

“Something Happens” \rightarrow “Continue Shower”

All-orders Probability that nothing happens

\[ \Delta(t_1, t_2) = \exp \left( - \int_{t_1}^{t_2} dt \frac{d\mathcal{P}}{dt} \right) \]  

(EXPONENTIALIZATION)

Analogous to nuclear decay

\[ N(t) \approx N(0) \exp(-ct) \]
1. Generate Random Number, $R \in [0,1]$

Solve equation $R = \Delta(t_1, t)$ for $t$ (with starting scale $t_1$)

Analytically for simple splitting kernels,
else numerically (or by trial+veto)
→ $t$ scale for next branching

2. Generate another Random Number, $R_z \in [0,1]$

To find second (linearly independent) phase-space invariant

Solve equation $R_z = \frac{I_z(z, t)}{I_z(z_{\text{max}}(t), t)}$ for $z$ (at scale $t$)

With the “primitive function”

$$I_z(z, t) = \int_{z_{\text{min}}(t)}^{z} dz \left. \frac{d\Delta(t')}{dt'} \right|_{t'=t}$$

3. Generate a third Random Number, $R_\varphi \in [0,1]$

Solve equation $R_\varphi = \varphi/2\pi$ for $\varphi$ → Can now do 3D branching
Perturbative Ambiguities

The final states generated by a shower algorithm will depend on

1. The choice of perturbative evolution variable(s) \( t^{[i]} \).
2. The choice of phase-space mapping \( d\Phi^{[i]}_{n+1}/d\Phi_n \).
3. The choice of radiation functions \( a_i \), as a function of the phase-space variables.
4. The choice of renormalization scale function \( \mu_R \).
5. Choices of starting and ending scales.

\( \rightarrow \) gives us additional handles for uncertainty estimates, beyond just \( \mu_R \)
(+ ambiguities can be reduced by including more pQCD \( \rightarrow \) matching!)
Nice to have all-orders solution

But it is only exact in the singular (soft & collinear) limits
→ gets the bulk of bremsstrahlung corrections right, but fails equally spectacularly: for hard wide-angle radiation: **visible, extra jets**

... which is exactly where fixed-order calculations work!

**So combine them!**
Summary: Parton Showers

Aim: generate events in as much detail as mother nature

→ Make stochastic choices ~ as in Nature (Q.M.) → Random numbers

**Factor** complete event probability into separate universal pieces, treated independently and/or sequentially (Markov-Chain MC)

Improve Born-level theory by including ‘most significant’ corrections

- Resonance decays (e.g., $t \rightarrow bW^+$, $W \rightarrow qq'$, $H^0 \rightarrow \gamma^0\gamma^0$, $Z^0 \rightarrow \mu^+\mu^-$, ...)
- Bremsstrahlung (FSR and ISR, exact in collinear and soft* limits)
- Hard radiation (matching)
- Hadronization (strings/clusters, discussed tomorrow)
- Additional Soft Physics: multiple parton-parton interactions, Bose-Einstein correlations, colour reconnections, hadron decays, ...

**Coherence***

Soft radiation → Angular ordering or Coherent Dipoles/Antennae
Matching
Example: $H^0 \rightarrow b\bar{b}$

**Born + Shower**

```
\begin{align*}
\text{Born + Shower} & & \text{Born + 1 @ LO} \\
\text{Diagram 1} & & \text{Diagram 2} \\
+ & & + \quad \ldots
\end{align*}
```

**Shower Approximation to Born + 1**
Example: $H^0 \rightarrow b\bar{b}$

**Born + Shower**

\[
\left(1 + g_s^2 2C_F \left[ \frac{2s_{ik}}{s_{ij}s_{jk}} + \frac{1}{s_{IK}} \left( \frac{s_{ij}}{s_{jk}} + \frac{s_{jk}}{s_{ij}} \right) \right] \right)^2 \]

**Born + 1 @ LO**

\[
\left(g_s^2 2C_F \left[ \frac{2s_{ik}}{s_{ij}s_{jk}} + \frac{1}{s_{IK}} \left( \frac{s_{ij}}{s_{jk}} + \frac{s_{jk}}{s_{ij}} + 2 \right) \right] \right)^2
\]

**Total Overkill** to add these two. All we really need is just that $+2 \ldots$
Adding Calculations

**Born × Shower**

\[ \begin{array}{cccc}
  X & X+1 & \ldots \\
  X & X+1 & X+2 & X+3 \\
  \text{Born} & X+1 & X+2 & X+3 \\
  \ldots & \ldots & \ldots & \ldots \\
\end{array} \]

**X+1 @ LO**

(with p_T cutoff, see previous lectures)

\[ \begin{array}{cccc}
  X+1 & \ldots \\
  X+1 & X+2 & X+3 & \ldots \\
  \text{X+1} & X+2 & X+3 & \ldots \\
  \ldots & \ldots & \ldots & \ldots \\
\end{array} \]

**Fixed-Order Matrix Element**

**Shower Approximation**

**Fixed-Order ME above p_T cut**

& **& **nothing below**
Adding Calculations

Born $\times$ Shower

\[
\begin{array}{cccc}
  X & X+1 & \cdots \\
  X & X+1 & X+2 & X+3 \\
  \text{Born} & X+1 & X+2 & X+3 \\
\end{array}
\]

Fixed-Order Matrix Element

Shower Approximation

$X+1$ @ LO $\times$ Shower

(with $p_T$ cutoff, see previous lectures)

\[
\begin{array}{cccc}
  X+1 & \cdots \\
  X+1 & X+2 & X+3 \\
  \text{Fixed-Order ME above $p_T$ cut} & \text{& nothing below} \\
\end{array}
\]

Shower approximation above $p_T$ cut & nothing below
Double Counting

Born $\times$ Shower + (X+1) $\times$ shower

Double Counting of terms present in both expansions

Worse than useless

Fixed-Order Matrix Element

Shower Approximation

Double counting above $p_T$ cut & shower approximation below
A (Complete Idiot’s) Solution – Combine

1. \([X]_{ME} + \text{showering}\)
2. \([X + 1 \text{ jet}]_{ME} + \text{showering}\)
3. ...

Doesn’t work

- \([X] + \text{shower} \text{ is inclusive}\)
- \([X+1] + \text{shower} \text{ is also inclusive}\)

Run generator for \(X (+ \text{shower})\)
Run generator for \(X+1 (+ \text{shower})\)
Run generator for … (+ shower)
Combine everything into one sample

What you get

\(\begin{align*}
\text{X inclusive} \\
\text{X+1 inclusive} \\
\text{X+2 inclusive}
\end{align*}\)

Overlapping “bins”

What you want

\(\begin{align*}
\text{X exclusive} \\
\text{X+1 exclusive} \\
\text{X+2 inclusive}
\end{align*}\)

One sample
First emission: “the HERWIG correction”

Use the fact that the angular-ordered HERWIG parton shower has a “dead zone” for hard wide-angle radiation (Seymour, 1995)

Many emissions: the MLM & CKKW-L prescriptions

(CKKW & Lönnblad, 2001) (Mangano, 2002) (+many more recent; see Alwall et al., EPJC53(2008)473)
Slicing: The Cost

1. Initialization time
   (to pre-compute cross sections and warm up phase-space grids)

2. Time to generate 1000 events
   (Z → partons, fully showered & matched. No hadronization.)

Z → n : Number of Matched Emissions

Z → udscb ; Hadronization OFF ; ISR OFF ; udsc MASSLESS ; b MASSIVE ; E_CM = 91.2 GeV ; Q_match = 5 GeV
SHERPA 1.4.0 (+COMIX) ; PYTHIA 8.1.65 ; VINCIA 1.0.29 (+MADGRAPH 4.4.26) ;
gcc/gfortran v 4.7.1 -O2 ; single 3.06 GHz core (4GB RAM)
The Gain

Example: W + Jets

Number of jets in pp → W+X at the LHC

From 0 (W inclusive) to W+3 jets

PYTHIA includes matching up to W+1 jet + shower

With ALPGEN, also the LO matrix elements for 2 and 3 jets are included (but Normalization still only LO)
Matching 2: Subtraction

LO × Shower

NLO

Examples: MC@NLO, aMC@NLO

Fixed-Order Matrix Element

Shower Approximation
Matching 2: Subtraction

LO × Shower

<table>
<thead>
<tr>
<th>X</th>
<th>X+1</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>X+1</td>
<td>X+2</td>
</tr>
<tr>
<td>Born</td>
<td>X+1</td>
<td>X+2</td>
</tr>
</tbody>
</table>

Fixed-Order Matrix Element

Shower Approximation

NLO - Shower_{NLO}

<table>
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</table>

Expand shower approximation to NLO analytically, then subtract:

Fixed-Order ME minus Shower Approximation (NOTE: can be < 0!)

Examples: MC@NLO, aMC@NLO
Matching 2: Subtraction

LO $\times$ Shower

(NLO - Shower\textsubscript{NLO}) $\times$ Shower

Fixed-Order Matrix Element

Shower Approximation

Fixed-Order ME minus Shower Approximation (NOTE: can be < 0!)

Subleading corrections generated by shower off subtracted ME

Examples: MC@NLO, aMC@NLO
Matching 2: Subtraction

Combine $\rightarrow$ MC@NLO

Consistent NLO + parton shower (though correction events can have $w<0$)
Recently, has been almost fully automated in aMC@NLO

Frederix, Frixione, Hirschi, Maltoni, Pittau, Torrielli, JHEP 1202 (2012) 048

Note 1: NOT NLO for $X+1$
Note 2: Multijet tree-level matching still superior for $X+2$

NB: $w < 0$ are a problem because they kill efficiency:
Extreme example: 1000 positive-weight - 999 negative-weight events $\rightarrow$ statistical precision of 1 event, for 2000 generated (for comparison, normal MC@NLO has ~ 10% neg-weights)
Matching 3: ME Corrections

Standard Paradigm:

Have ME for X, X+1, ..., X+n;
Want to combine and add showers \[ \rightarrow \text{“The Soft Stuff”} \]

Works pretty well at low multiplicities

Still, only corrected for “hard” scales; Soft still pure LL.

At high multiplicities:

Efficiency problems: slowdown from need to compute and generate phase space from \[ d\sigma_{X+n} \], and from unweighting (efficiency also reduced by negative weights, if present)

Scale hierarchies: smaller single-scale phase-space region

Powers of alphaS pile up

Better Starting Point: a QCD fractal?
Interleaved Paradigm:

Have shower; want to improve it using ME for $X, X+1, \ldots, X+n$.

Interpret all-orders shower structure as a “trial distribution”

**Quasi-scale-invariant**: intrinsically multi-scale (resums logs)

**Unitary**: automatically unweighted (& IR divergences $\rightarrow$ multiplicities)

More precise expressions imprinted via veto algorithm: ME corrections at LO, NLO, ... $\rightarrow$ soft and hard corrections

No additional phase-space generator or $\sigma_{X+n}$ calculations $\rightarrow$ fast

**+ Can get Automated Theory Uncertainties**

For each event: vector of output weights (central value = 1)

+ Uncertainty variations. Faster than $N$ separate samples; only one sample to analyse, pass through detector simulations, etc.

**LO**: Giele, Kosower, Skands, *PRD84*(2011)054003

**NLO**: Hartgring, Laenen, Skands, *arXiv:1303.4974*
Matching 3: ME Corrections

Examples: PYTHIA, POWHEG, VINCIA

- **Start at Born level**
  \[ |M_F|^2 \]

- **Generate “shower” emission**
  \[ |M_{F+1}|^2 \overset{LL}{\sim} \sum_{i \in \text{ant}} a_i |M_F|^2 \]

- **Correct to Matrix Element**
  \[ a_i \rightarrow \frac{|M_{F+1}|^2}{\sum a_i |M_F|^2} a_i \]

- **Unitarity of Shower**
  Virtual = \(- \int \text{Real} \)

- **Correct to Matrix Element**
  \[ |M_F|^2 \rightarrow |M_F|^2 + 2\text{Re}[M_F^1 M_F^0] + \int \text{Real} \]

**Virtues:**
- No “matching scale”
- No negative-weight events
- Can be very fast

**First Order**
- **PYTHIA**: LO\(_1\) corrections to most SM and BSM decay processes, and for pp \(\rightarrow Z/W/H\) (Sjöstrand 1987)
- **POWHEG** (& POWHEG BOX): LO\(_1\) + NLO\(_0\) corrections for generic processes (Frixione, Nason, Oleari, 2007)

**Multileg NLO:**
- **VINCIA**: LO\(_{1,2,3,4}\) + NLO\(_{0,1}\) (shower plugin to PYTHIA 8; formalism for pp soon to appear) (see previous slide)
- **MiNLO**-merged POWHEG: LO\(_{1,2}\) + NLO\(_{0,1}\) for pp \(\rightarrow Z/W/H\)
- **UNLOPS**: for generic processes (in PYTHIA 8, based on POWHEG input) (Lönnblad & Prestel, 2013)

Illustrations from: PS, TASI Lectures, arXiv:1207.2389
Larkoski, Lopez-Villarejo, Skands, PRD 87 (2013) 054033

1. Initialization time
(to pre-compute cross sections and warm up phase-space grids)

2. Time to generate 1000 events
(Z → partons, fully showered & matched. No hadronization.)

Z→n : Number of Matched Legs

Z→udscb ; Hadronization OFF ; ISR OFF ; udsc MASSLESS ; b MASSIVE ; E_CM = 91.2 GeV ; Q_match = 5 GeV
SHERPA 1.4.0 (+COMIX) ; PYTHIA 8.1.65 ; VINCIA 1.0.29 + MADGRAPH 4.4.26 ;
gcc/gfortran v 4.7.1 -O2 ; single 3.06 GHz core (4GB RAM)
Summary: Two ways to compute Quantum Corrections

Fixed Order: consider a specific physical process

Explicit solutions (to given perturbative order)
- Standard-Model: typically NLO or NNLO
- Beyond-SM: typically LO or NLO

Limited generality

Event generators: consider all possible physical processes (within perturbative QFT)

Approximate solutions
- Process-dependence = subleading correction (→ matching)

Maximum generality
- Emphasis is on universalities; physics
- Common property of all processes is, eg, the limits in which they factorize!

Increasingly, the gold standard is calculations that combine the best of both worlds!
These are, however, subtle, and the structure of the perturbative series remains intriguing
Simple Monte Carlo Example: Number of AEPSHEP students who will get hit by a car this week

Complicated Function:

- **Time-dependent**
  - Traffic density during day, week-days vs week-ends
  - (i.e., non-trivial time evolution of system)

- **No two students are the same**
  - Need to compute probability for each and sum
  - (simulates having several distinct types of “evolvers”)

- **Multiple outcomes:**
  - Hit → keep walking, or go to hospital?
  - Multiple hits = Product of single hits, or more complicated?
Monte Carlo Approach

Approximate Traffic

Simple overestimate:

- highest recorded density
- of most careless drivers,
- driving at highest recorded speed
...

Approximate Student

by most completely reckless and accident-prone student
(wandering the streets lost in thought after these lectures …)

This extreme guess will be the equivalent of our simple overestimate from yesterday:
Off we go…

Throw random accidents according to:

\[
R = \int_{t_0}^{t_e} dt \int_x dx \sum_{i=1}^{n_{\text{stud}}} \alpha_i(x, t) \rho_i(x, t) \rho_c(x, t)
\]

\[
R = (t_e - t_0) \Delta x \alpha_{\max} n_{\text{stud}} \rho_{c_{\max}}
\]

(Also generate trial \(x_e\), e.g., uniformly in circle around Puri)

(Also generate trial \(i\); a random student gets hit)
Accept trial hit \((i,x,t)\) with probability

\[
\text{Prob(accept)} = \frac{\alpha_i(x,t) \rho_i(x,t) \rho_c(x,t)}{\alpha_{\text{max}} n_{\text{stud}} \rho_{\text{cmax}}}
\]

\(\alpha_i(x,t)\): The actual “hit rate” (OK, not really known, but can make one up)

\(\rho_c\): actual density of cars at location \(x\) at time \(t\)
\(\rho_i\): actual density of student \(i\) at location \(x\) at time \(t\)

Using the following:

\(\rightarrow\) True number = number of accepted hits
(note: we didn’t really treat multiple hits … \(\rightarrow\) Markov Chain)
Evolution

\[ Q \sim Q_x \]

- **Leading Order**
  - % of LO
  - Born, +1, +2

- **“Experiment”**
  - % of \( \sigma_{\text{tot}} \)
  - Born (exc), +1 (exc), +2 (inc)

Exclusive = n and only n jets
Inclusive = n or more jets
Evolution

\[ Q \sim \frac{Q_x}{\text{"A few"}} \]

"Experiment"

% of \( \sigma_{\text{tot}} \)

% of LO

Leading Order

- Born
- +1
- +2

Born (exc)
+1 (exc)
+2 (inc)

Exclusive = n and only n jets
Inclusive = n or more jets
Evolution

\[ Q \ll Q_X \]

- Cross Section Diverges
- Cross Section Remains = Total (IR safe)
- Number of Partons Diverges (IR unsafe)

**UNITARITY**
Jet clustering algorithms

Map event from low E-resolution scale (i.e., with many partons/hadrons, most of which are soft) to a higher E-resolution scale (with fewer, hard, IR-safe, jets)

Parton shower algorithms

Map a few hard partons to many softer ones

Probabilistic → closer to nature.

Not uniquely invertible by any jet algorithm*

---

Jet Clustering
(Deterministic*)
(Winner-takes-all)

Parton Showering
(Probabilistic)

Born-level ME

Q ~ Λ ~ mπ
~ 150 MeV

Q ~ Q_{had}
~ 1 GeV

Q ~ E_{cm}
~ M_X

*) See “Qjets” for a probabilistic jet algorithm, arXiv:1201.1914

*) See “Sector Showers” for a deterministic shower, arXiv:1109.3608
Choice of slicing scale (=matching scale)

Fixed order must still be reliable when regulated with this scale

→ matching scale should never be chosen more than ~ one order of magnitude below hard scale.

Precision still “only” Leading Order

Choice of Renormalization Scale

We already saw this can be very important (and tricky) in multi-scale problems.

Caution advised (see also supplementary slides & lecture notes)
A scale of 20 GeV for a W boson becomes 40 GeV for something weighing $2M_W$, etc … (+ adjust for $C_A/C_F$ if g-initiated)

The matching scale should be written as a ratio (Bjorken scaling)
Using a too low matching scale → everything just becomes highest ME

*Caveat emptor: showers generally do not include helicity correlations*
Uncertainty Estimates

a) Authors provide specific “tune variations”
   Run once for each variation → envelope


b) One shower run
   + unitarity-based uncertainties → envelope

Giele, Kosower, PS; Phys. Rev. D84 (2011) 054003
Uncertainty Estimates

a) Authors provide specific “tune variations”  
Run once for each variation → envelope

![Graph showing 1/N dN/d(1-T) vs 1-T](image)


Vincia:uncertaintyBands = on

PYTHIA 6 example
Perugia Variations

μR

Ratio to ALEPH

Plot from mcplots.cern.ch

b) **One shower run**  
+ unitarity-based uncertainties → envelope

![Graph showing 1/N dN/d(1-T) vs 1-T](image)

Giele, Kosower, PS; Phys. Rev. D84 (2011) 054003

Vincia 1.027 + MadGraph 4.426 + Pythia 8.153

Data from Phys.Rept. 399 (2004) 71

Matching reduces uncertainty
Precision on integral dominated by the points with $f \approx f_{\text{max}}$ (i.e., peak regions)

$\rightarrow$ slow convergence if high, narrow peaks
Stratified Sampling

→ Make it twice as likely to throw points in the peak

Choose:

- [0,1] → Region A
- [1,2] → Region B
- 6*R₁ ∈ [2,4] → Region C
- [4,5] → Region D
- [5,6] → Region E

→ faster convergence for same number of function evaluations
Adaptive Sampling

→ Can even design algorithms to do this automatically as they run (not covered here)

→ Adaptive sampling
Importance Sampling

→ or throw points according to some smooth peaked function for which you have, or can construct, a random number generator (here: Gauss)

E.g., VEGAS algorithm, by G. Lepage
Why does this work?

1) You are inputting knowledge: obviously need to know where the peaks are to begin with … (say you know, e.g., the location and width of a resonance)

2) Stratified sampling increases efficiency by combining fixed-grid methods with the MC method, with further gains from adaptation

3) Importance sampling:

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{b} \frac{f(x)}{g(x)} \, dG(x)
\]

Effectively does flat MC with changed integration variables

Fast convergence if \( f(x)/g(x) \approx 1 \)
How we do Monte Carlo

Take your system

- Set of radioactive nuclei
- Set of hard scattering processes
- Set of resonances that are going to decay
- Set of particles coming into your detector
- Set of cosmic photons traveling across the galaxy
- Set of molecules

...
How we do Monte Carlo

Take your system

Generate a “trial” (event/decay/interaction/…)

- Not easy to generate random numbers distributed according to exactly the right distribution?
- May have complicated dynamics, interactions …
- → use a simpler “trial” distribution

Flat with some stratification

Or importance sample with simple overestimating function (for which you can generate random #s)
Take your system

Generate a “trial” (event/decay/interaction/… )

Accept trial with probability \( f(x)/g(x) \)

- \( f(x) \) contains all the complicated dynamics
- \( g(x) \) is the simple trial function

If accept: replace with new system state
If reject: keep previous system state

And keep going: generate next trial …
How we do Monte Carlo

Take your system

Generate a “trial” (event/decay/interaction/…)

Accept trial with probability \( \frac{f(x)}{g(x)} \)

- \( f(x) \) contains all the complicated dynamics
- \( g(x) \) is the simple trial function

If accept: replace with new system state
If reject: keep previous system state

And keep going:

- no dependence on \( g \) in final result - only affects convergence rate

Sounds deceptively simple, but …

with it, you can integrate arbitrarily complicated functions (in particular chains of nested functions), over arbitrarily complicated regions, in arbitrarily many dimensions …