QCD and Monte Carlos

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Lecture Notes: P. Skands, arXiv:1207.2389
Recap: Quantum Field Theory

The **elementary** interactions are encoded in the **Lagrangian**

QFT $\rightarrow$ Feynman Diagrams $\rightarrow$ Perturbative Expansions (in $\alpha_s$)

$g_s^2 = 4\pi\alpha_s$

**THE BASIC ELEMENTS OF QCD: QUARKS AND GLUONS**

$\psi_q^j = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$

The Lagrangian of QCD

$$L = \bar{\psi}_q^i (i\gamma^\mu) (D_\mu)_{ij} \psi_q^j - m_q \bar{\psi}_q^i \psi_q^i - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}$$

$D_{\mu ij} = \delta_{ij} \partial_\mu - ig_s T^a_{ij} A^a_\mu$

Gauge Covariant Derivative: makes $L$ invariant under SU(3)$_C$ rotations of $\psi_q$

$m_q$: Quark Mass Terms

(Higgs + QCD condensates)

Gluon-Field Kinetic Terms and Self-Interactions

$$F_{\mu\nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu$$
QCD is more than just a perturbative expansion in $\alpha_s$.

The relation between $\alpha_s$, Feynman diagrams, and the full QCD dynamics is under active investigation. Emergent phenomena:

**Jets** (the fractal of perturbative QCD) $\leftrightarrow$ amplitude structures in quantum field theory $\leftrightarrow$ factorisation & unitarity. Precision jet (structure) studies.

**Strings** (strong gluon fields) $\leftrightarrow$ quantum-classical correspondence. String physics. String breaks. Dynamics of hadronization phase transition.

**Hadrons** $\leftrightarrow$ Spectroscopy (incl excited and exotic states), lattice QCD, (rare) decays, mixing, light nuclei. Hadron beams $\rightarrow$ multiparton interactions, diffraction, …
The Lagrangian of QCD

\[ \mathcal{L} = \bar{\psi}^i_q (i\gamma^\mu) (D_\mu)_{ij} \psi^j_q - m_q \bar{\psi}^i_q \psi^i_q - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \]

+ ... ... ... ...

LHC Run 1: still no explicit "new physics"

→ we’re still looking for deviations from SM

Accurate modeling of QCD improve searches & precision

There are more things in heaven and earth, Horatio, than are dreamt of in your philosophy.

Hamlet.
High-cross section physics

Total $\sigma_{pp} \sim 100 \text{ mb} = 10^{11} \text{ pb}$

$\sigma_{EW} \sim 10^8 \text{ fb} = 10^5 \text{ pb}$
• 1st jet: $p_T = 520$ GeV, $\eta = -1.4$, $\phi = -2.0$
• 2nd jet: $p_T = 460$ GeV, $\eta = 2.2$, $\phi = 1.0$
• 3rd jet: $p_T = 130$ GeV, $\eta = 0.3$, $\phi = 1.2$
• 4th jet: $p_T = 50$ GeV, $\eta = -1.0$, $\phi = -2.9
Let's start by considering some of the basic ingredients of calculations for processes with QCD jets (partons).
Interactions in Colour Space

Colour Factors

All QCD processes have a “colour factor”. It counts the enhancement from the sum over colours.

\[
\sum_{\text{colours}} |M|^2 = \text{Diagram}
\]

\[
i,j \in \{R, G, B\}
\]
Interactions in Colour Space

Colour Factors

All QCD processes have a “colour factor”. It counts the enhancement from the sum over colours.

\[
\sum_{\text{colours}} |M|^2 = \delta_{ij} \quad \propto \delta_{ij} \delta_{ji} = \text{Tr}[\delta_{ij}] = N_C
\]

\[i, j \in \{R, G, B\}\]
Interactions in Colour Space

Colour Factors

All QCD processes have a “colour factor”. It counts the enhancement from the sum over colours.

\[
\sum_{\text{colours}} |M|^2 = \delta_{ij} \propto \delta_{ij} \delta_{ji} = \text{Tr}[\delta_{ij}] = NC \]

(Drell & Yan, 1970)
Interactions in Colour Space

Colour Factors

All QCD processes have a “colour factor”. It counts the enhancement/suppression from the sum/average over colours.

\[
\frac{1}{9} \sum_{\text{colours}} |M|^2 = \delta_{ij} \qquad \text{Drell-Yan}
\]

\[
\delta_{ij} \propto \delta_{ij} \delta_{ji} \frac{1}{N_C^2} = \text{Tr}[\delta_{ij}] \frac{1}{N_C^2} = 1/N_C
\]

\(i,j \in \{R,G,B\}\)

(Drell & Yan, 1970)
Interactions in Colour Space

Colour Factors

All QCD processes have a “colour factor”. It counts the enhancement/suppression from the sum/average over colours.

\[
\sum_{\text{colours}} |M|^2 = \delta_{ij} T_{a}^{jk} g_{a}^{jk} \rightarrow q_i q_j q_k \\
\delta_{il} T_{a}^{\ell k} g_{a}^{\ell k} \rightarrow q_i q_{\ell} q_k
\]

\[
\propto \delta_{ij} T_{a}^{jk} (T_{a}^{\ell k} \delta_{i\ell})^* \\
= \text{Tr}[T_{a} T_{a}] \\
= \frac{1}{2} \text{Tr}\delta_{ab} \\
= 4
\]
Colour factors (squared) produce traces

\[
\text{Tr}(t^A t^B) = T_R \delta^{AB}, \quad T_R = \frac{1}{2}
\]

(Example Diagram)

(from ESHEP lectures by G. Salam)
Quick Guide to Colour Algebra

Colour factors (squared) produce traces

\[ \text{Trace Relation} \]

\[ \text{Example Diagram} \]

\[ \text{Relation} \]

\[ \text{Example Diagram} \]

\[ \text{Relation} \]

\[ \text{Example Diagram} \]

\[ \text{Relation} \]

\[ \text{Example Diagram} \]

\[ \text{Relation} \]

\[ \text{Example Diagram} \]
Quick Guide to Colour Algebra

Colour factors (squared) produce traces

\[ \text{Tr}(t^A t^B) = T_R \delta^{AB}, \quad T_R = \frac{1}{2} \]

\[ \sum_A t^A_{ab} t^A_{bc} = C_F \delta_{ac}, \quad C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3} \]

\[ \sum_{C,D} f^{ACD} f^{BCD} = C_A \delta^{AB}, \quad C_A = N_c = 3 \]

\[ t^A_{ab} t^A_{cd} = \frac{1}{2} \delta_{bc} \delta_{ad} - \frac{1}{2N_c} \delta_{ab} \delta_{cd} \quad \text{(Fierz)} \]

(from ESHEP lectures by G. Salam)
Gluon-Gluon Interactions

\[ \mathcal{L} = \overline{\psi}_q (i \gamma^\mu) (D_\mu)_{ij} \psi^i_q - m_q \overline{\psi}_q \psi_q \frac{1}{4} F^a_{\mu \nu} F^{a \mu \nu} \]

Gluon field strength tensor:

\[ F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu \]

Structure constants of SU(3):

- \( f_{123} = 1 \)
- \( f_{147} = f_{246} = f_{257} = f_{345} = \frac{1}{2} \)
- \( f_{156} = f_{367} = -\frac{1}{2} \)
- \( f_{458} = f_{678} = \frac{\sqrt{3}}{2} \)

Antisymmetric in all indices

All other \( f_{ijk} = 0 \)
Digression: Colour Interference

In general, many different diagrams will contribute to each process, with different colour structures, e.g.:

→ diagrams squared

+ quantum interferences

If this was all: could define a positive definite probability for each colour structure ~ “LC”

“non-planar”

Mixed signs, do not correspond to a unique colour structure (squared) ~ “Subleading Colour”; hard to treat in MCs!
MC generators use a set of simple rules for color flow, based on large-$N_C$ limit.  

\[(Never \ Twice \ Same \ Color: \ true \ up \ to \ O(1/N_C^2))\]

\[q \rightarrow qg\]

\[g \rightarrow q\bar{q}\]

\[g \rightarrow gg\]

→ a system of “colour dipoles”

+ Inside each dipole, interference effects can be included (coherence, more later)

Also tells us between which partons confining potentials will arise (more in lecture 3)
For an entire Cascade

Example: $Z^0 \rightarrow qq$

Coherence of pQCD cascades $\rightarrow$ not much "overlap" between singlet subsystems
\rightarrow Leading-colour approximation pretty good

LEP measurements in WW confirm this (at least to order $10\% \sim 1/N_c^2$)

Note: (much) more color getting kicked around in hadron collisions $\rightarrow$ more later
QCD at Fixed Order

Distribution of observable: $\mathcal{O}$

In production of $X + \text{anything}$

\[
\left. \frac{d\sigma}{d\mathcal{O}} \right|_{\text{ME}} = \sum_{k=0}^{\infty} \int d\Phi_{X+k} \left| \sum_{\ell=0}^{\infty} M_{X+k}^{(\ell)} \right|^2 \delta (\mathcal{O} - \mathcal{O}(\{p\}_{X+k}))
\]

Fixed Order (All Orders)

Cross Section differentially in $\mathcal{O}$

Sum over "anything" $\approx$ legs

Phase Space

Sum over identical amplitudes, then square

Matrix Elements for $X+k$ at ($\ell$) loops

Momentum configuration

Evaluate observable $\rightarrow$ differential in $\mathcal{O}$

Truncate at $k = 0, \ell = 0$,

$\rightarrow$ Born Level = First Term

Lowest order at which $X$ happens
Another representation

Truncate at $k = 0, \ell = 0$, → Born Level
Lowest order at which X happens
Loops and Legs

Note: \((X+1)\)-jet observables only correct at LO
Loops and Legs

Note: X+2 jet observables only correct at LO

Note: X+1 jet observables only correct at NLO

Figure 14: Coefficients of the perturbative series covered by an NNLO calculation. The total power of $s$ for each coefficient is $n = k + 1$. Green shading represents the full perturbative coefficient at the respective $k$ and $\ell$.

\[ X @ \text{NNLO} \]
(includes X+1 @ NLO)
(includes X+2 @ LO)

To extend the integration to cover also the case of 2 unresolved jets, we must combine the left- and right-hand parts of figure 13 and add the new coefficient

\[ \sigma^{(2)}_{0} = |M^{(1)}_{0}|^{2} + 2\text{Re}[M^{(2)}_{0}M^{(0)}_{\ast}], \]

as illustrated by the diagram in figure 14.

2.4 The Subtraction Idea

According to the KLN theorem, the IR singularities coming from integrating over collinear and soft real-emission configurations should cancel, order by order, by those coming from the IR divergent loop integrals. This implies that we should be able to rewrite e.g. the NLO cross section, equation (47), as

\[ \text{NLO} = \text{Born} + \text{Finite} \]

\[ \Rightarrow Z \int F_{\ell}^{1} |M^{(0)}_{\ell+1}|^{2} + F_{\text{Finite}} \]

\[ \text{NLO} = \text{Finite} \]

with the second and third terms having had their common (but opposite-sign) singularities canceled out and some explicitly finite quantities remaining.
Cross sections at LO

**Born @ LO**

\[ \sigma_{\text{Born}} = \int |M_X^{(0)}|^2 \]

**Born + n @ LO**

\[ \sigma_{X+1}^{\text{LO}}(R) = \int_R |M_{X+1}^{(0)}|^2 \]

Infrared divergent \( \rightarrow \) **Must be regulated**

\[ R = \text{some Infrared Safe phase space region} \]

(Often a cut on \( p_\perp > n \) GeV)

**Careful not to take it too low!**
The Infrared Strikes Back

Naively, QCD radiation suppressed by $\alpha_s \approx 0.1$

Truncate at fixed order = LO, NLO, …

E.g., $\sigma(X+\text{jet})/\sigma(X) \propto \alpha_s$

**Example:** Pair production of SUSY particles at LHC$_{14}$, with $M_{\text{SUSY}} \approx 600$ GeV

<table>
<thead>
<tr>
<th>FIXED ORDER pQCD</th>
<th>$\sigma_{\text{tot}}$[pb]</th>
<th>$\tilde{g}\tilde{g}$</th>
<th>$\tilde{u}_L\tilde{g}$</th>
<th>$\tilde{u}_L\tilde{u}_L^*$</th>
<th>$\tilde{u}_L\tilde{\bar{u}}_L$</th>
<th>$T\bar{T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_T,j &gt; 100$ GeV</td>
<td>$\sigma_{0j}$</td>
<td>4.83</td>
<td>5.65</td>
<td>0.286</td>
<td>0.502</td>
<td>1.30</td>
</tr>
<tr>
<td>inclusive $X + 1$ “jet”</td>
<td>$\sigma_{1j}$</td>
<td>2.89</td>
<td>2.74</td>
<td>0.136</td>
<td>0.145</td>
<td>0.73</td>
</tr>
<tr>
<td>inclusive $X + 2$ “jets”</td>
<td>$\sigma_{2j}$</td>
<td>1.09</td>
<td>0.85</td>
<td>0.049</td>
<td>0.039</td>
<td>0.26</td>
</tr>
</tbody>
</table>

| $p_T,j > 50$ GeV | $\sigma_{0j}$   | 4.83            | 5.65           | 0.286         | 0.502          | 1.30      |
| $\sigma_{1j}$ | 5.90            | 5.37           | 0.283         | 0.283        | 1.50           |
| $\sigma_{2j}$ | 4.17            | 3.18           | 0.179         | 0.117        | 1.21           |

$\sigma_{50} \sim \sigma_{\text{tot}}$ tells us that there will “always” be a $\sim 50$-GeV jet “inside” a 600-GeV process

All the scales are high, $Q >> 1$ GeV, so perturbation theory **should** be OK …
The Lagrangian of QCD is **scale invariant**

(neglecting small quark masses)

Characteristic of point-like constituents

To first approximation, observables depend only on dimensionless quantities, like **angles** and energy **ratios**

Also means that when we look closer, patrons (quarks and gluons) must generate ever self-similar patterns = **fractals**

Jets-within-jets-within-jets …

Note: scaling **violation** is induced in full QCD, but only by renormalization: \( g_s^2 = 4\pi\alpha_s(\mu) \)
(some) Physics

Charges Stopped, kicked, or created

The harder they stop, the harder the fluctuations that continue to become radiation

Radiation

Radiation

a.k.a. Bremsstrahlung
Synchrotron Radiation

cf. equivalent-photon approximation
Weiszäcker, Williams
~ 1934

cf. equivalent-photon approximation
Weiszäcker, Williams
~ 1934
Jets \cong \text{Fractals}

- **Most bremsstrahlung** is driven by divergent propagators \( \rightarrow \) simple structure.

- **Amplitudes factorize in singular limits** (\( \rightarrow \) universal “conformal” or “fractal” structure).

Partons \( ab \) \( \rightarrow \) “collinear”:

\[
|\mathcal{M}_{F+1}(\ldots, a, b, \ldots)|^2 \overset{\text{ab}}{\to} |\mathcal{M}_F(\ldots, a + b, \ldots)|^2
\]

\[
P(z) = \text{DGLAP splitting kernels, with } z = \text{energy fraction } = \frac{E_a}{E_a + E_b}
\]

Gluon \( j \) \( \rightarrow \) “soft”:

\[
|\mathcal{M}_{F+1}(\ldots, i, j, k \ldots)|^2 \overset{jg}{\to} |\mathcal{M}_F(\ldots, i, k, \ldots)|^2
\]

\[
g_s^2 \to 4\pi \alpha_s(Q^2)
\]

Coherence \( \rightarrow \) Parton \( j \) really emitted by \( (i,k) \) “colour antenna” (in leading colour approximation).

Can apply this many times \( \rightarrow \) nested factorizations

Jets-within-jets-within-jets … \( \rightarrow \) lecture on showers.
Lessons:

• Each time we add a QCD parton, we get singularities
• Driven by intermediate propagators going “on shell”
• They are \textit{universal} (process-independent) and imply that, in the singular limits (soft/collinear), QCD amplitudes factorize.

But then don’t we get infinite cross sections? And what about when we add loops?
Cross sections at NLO

NLO:

\[ \sigma_{NLO}^X = \int |M_X^{(0)}|^2 + \int |M_X^{(0)}|^2 + \int 2\text{Re}[M_X^{(1)}M_X^{(0)*}] \]

(note: this is not the 1-loop diagram squared)

KLN Theorem (Kinoshita-Lee-Nauenberg)

Sum over ‘degenerate quantum states’: Singularity cancel at complete order (only finite terms left over)

\[ = \sigma_{\text{Born}+\text{Finite}} \left\{ \int |M_X^{(0)}|^2 \right\} + \text{Finite} \left\{ \int 2\text{Re}[M_X^{(1)}M_X^{(0)*}] \right\} \]

\[ \sigma_{NLO}(e^+e^- \rightarrow q\bar{q}) = \sigma_{LO}(e^+e^- \rightarrow q\bar{q}) \left( 1 + \frac{\alpha_s(E_{\text{CM}})}{\pi} + O(\alpha_s^2) \right) \]
The Subtraction Idea

How do I get finite\{Real\} and finite\{Virtual\}? 
First step: classify IR singularities using universal functions

EXAMPLE: factorization of amplitudes in the soft limit

\[
|\mathcal{M}_{n+1}(1, \cdots, i, j, k, \cdots, n + 1)|^2 \xrightarrow{j \rightarrow 0} g_s^2 C_{ijk} S_{ijk} |\mathcal{M}_n(1, \cdots, i, k, \cdots, n + 1)|^2
\]

Universal “Soft Eikonal”

\[
S_{ijk}(m_I, m_K) = \frac{2s_{ik}}{s_{ij}s_{jk}} - \frac{2m_I^2}{s_{ij}^2} - \frac{2m_K^2}{s_{jk}^2}, \quad s_{ij} \equiv 2p_i \cdot p_j
\]
The Subtraction Idea

Add and subtract IR limits (SOFT and COLLINEAR)

\[
d\sigma_{NLO} = \int d\Phi_{m+1} \left( d\sigma^R_{NLO} - d\sigma^S_{NLO} \right) + \left[ \int d\Phi_{m+1} d\sigma^S_{NLO} + \int d\Phi_m d\sigma^V_{NLO} \right]
\]

Finite by Universality

Finite by KLN

Choice of subtraction terms:

Singularities mandated by gauge theory

Non-singular terms: up to you (added and subtracted, so vanish)

\[
\frac{|\mathcal{M}(Z^0 \rightarrow q_i g_j \bar{q}_k)|^2}{|\mathcal{M}(Z^0 \rightarrow q_I \bar{q}_K)|^2} = g_s^2 2C_F \left[ \frac{2s_{ik}}{s_{ij} s_{jk}} + \frac{1}{s_{IK}} \left( \frac{s_{ij}}{s_{jk}} + \frac{s_{jk}}{s_{ij}} \right) \right]
\]

SOFT

\[
\frac{|\mathcal{M}(H^0 \rightarrow q_i g_j \bar{q}_k)|^2}{|\mathcal{M}(H^0 \rightarrow q_I \bar{q}_K)|^2} = g_s^2 2C_F \left[ \frac{2s_{ik}}{s_{ij} s_{jk}} + \frac{1}{s_{IK}} \left( \frac{s_{ij}}{s_{jk}} + \frac{s_{jk}}{s_{ij}} + 2 \right) \right]
\]

COLLINEAR

Dipoles (Catani-Seymour)

Global Antennae (Gehrmann, Gehrmann-de Ridder, Glover)

Sector Antennae (Kosower)

\[\vdots\]
Structure of $\sigma(\text{NNLO})$

**NNLO**

\[
\sigma_{\text{NNLO}}^{X} = \sigma_{X}^{\text{NLO}} + \int \left( |M_{X}^{(1)}|^2 + 2\text{Re}[M_{X}^{(2)}M_{X}^{(0)*}] \right) + \int 2\text{Re}[M_{X+1}^{(1)}M_{X+1}^{(0)*}] + \int |M_{X+2}^{(0)}|^2
\]

**I-Loop × I-Loop**

**I-Loop × Real (X+1)**

**Born × Real (X+2)**

**Two-Loop × Born Interference**
Infrared Safety

**Definition:** an observable is **infrared safe** if it is **insensitive to**

**SOFT radiation:**
Adding any number of infinitely **soft** particles (zero-energy) should not change the value of the observable

**COLLINEAR radiation:**
Splitting an existing particle up into two **comoving** ones (conserving the total momentum and energy) should not change the value of the observable

Note: some people use the word “infrared” to refer to soft only. Hence you may also hear “infrared and collinear safety”. Advice: always be explicit and clear what you mean.
Consequences of Collinear Unsafety

Collinear Safe

\[ \alpha_s^n \times (-\infty) \]

Collinear Unsafe

\[ \alpha_s^n \times (+\infty) \]

Infinities cancel

(KLN: ‘degenerate states’)

Infinities do not cancel

Invalidates perturbation theory

Real life does not have infinities, but pert. infinity leaves a real-life trace

\[ \alpha_s^2 + \alpha_s^3 + \alpha_s^4 \times \infty \rightarrow \alpha_s^2 + \alpha_s^3 + \alpha_s^4 \times \ln p_t/\Lambda \rightarrow \alpha_s^2 + \alpha_s^3 + \alpha_s^3 \]

BOTH WASTED
Lessons: “Stereo Vision”

Use IR Safe algorithms
To study short-distance physics
These days, ≈ as fast as IR unsafe algorithms and widely implemented (e.g., FASTJET), including

“Cone-like”: SiS Cone, Anti-\(k_T\), …
“Recombination-like”: \(k_T\), Cambridge/Aachen, Anti-\(k_T\), …

http://www.fastjet.fr/

Use IR Sensitive observables
E.g., number of tracks, identified particles, …
To explicitly study hadronization and check models of IR physics

More about IR in lecture on soft QCD …
Hadrons are composite, with time-dependent structure:

Partons within clouds of further partons, constantly emitted and absorbed

For hadron to remain intact, virtualities $k^2 < M_h^2$

High-virtuality fluctuations suppressed by powers of

$$\frac{\alpha_s M_h^2}{k^2}$$

$M_h$ : mass of hadron
$k^2$ : virtuality of fluctuation

$\rightarrow$ Lifetime of fluctuations $\sim 1/M_h$

Hard incoming probe interacts over much shorter time scale $\sim 1/Q$

On that timescale, partons $\sim$ frozen

Hard scattering knows nothing of the target hadron apart from the fact that it contained the struck parton $\rightarrow$ factorisation

Illustration from T. Sjöstrand
In DIS, there is a formal proof of factorization

(Collins, Soper, 1987)

Deep Inelastic Scattering (DIS)

Surprise Question: What’s the color factor for DIS?

We really can write the cross section in factorized form:

\[
\sigma^{\ell h} = \sum_i \sum_f \int dx_i \int d\Phi_f f_{i/h}(x_i, Q_F^2) \frac{d\sigma^{\ell i \rightarrow f}(x_i, \Phi_f, Q_F^2)}{dx_i d\Phi_f}
\]

\[\Phi_f = \text{Final-state phase space}\]
\[f_{i/h} = \text{PDFs}\]
\[Q^2 = Q_F^2\]

Differential partonic Hard-scattering Matrix Element(s)
A propos Factorization

Why do we need PDFs, parton showers / jets, etc.? Why are Fixed-Order QCD matrix elements not enough?

F.O. QCD requires **Large scales** : to guarantee that $\alpha_s$ is small enough to be perturbative (not too bad, since we anyway *often* want to consider large-scale processes [insert your fav one here])

F.O. QCD requires **No hierarchies** : conformal structure implies that soft/collinear hierarchies are associated with on-shell singularities that ruin fixed-order expansion.

**But!!!** we collide - and observe - low-scale hadrons, with *non-perturbative structure*, that participate in hard processes, whose scales are *hierarchically greater* than $m_{\text{had}} \sim 1 \text{ GeV}$.

→ A Priori, no perturbatively calculable observables in QCD
Lesson: Factorization $\rightarrow$ can still calculate!

Why is Fixed Order QCD not enough?
: It requires all resolved scales $>> \Lambda_{\text{QCD}}$ AND no large hierarchies

**PDFs:** connect incoming hadrons with the high-scale process

**Fragmentation Functions:** connect high-scale process with final-state hadrons
(each is a non-perturbative function modulated by initial- and final-state radiation)

\[
\frac{d\sigma}{dX} = \sum_{a,b} \sum_f \int \hat{X}_f \, f_a(x_a, Q_i^2) f_b(x_b, Q_i^2) \frac{d\hat{\sigma}_{ab \rightarrow f}(x_a, x_b, f, Q_i^2, Q_f^2)}{d\hat{X}_f} D(\hat{X}_f \rightarrow X, Q_i^2, Q_f^2)
\]

**PDFs:** needed to compute inclusive cross sections

**FFs:** needed to compute (semi-)exclusive cross sections

**Resummed pQCD:** All resolved scales $>> \Lambda_{\text{QCD}}$ AND $X$ Infrared Safe

*)pQCD = perturbative QCD

Will take a closer look at parton showers in the next lecture
Real QCD isn’t conformal

The coupling runs logarithmically with the energy scale

\[ Q^2 \frac{\partial \alpha_s}{\partial Q^2} = \beta(\alpha_s) \]

\[ \beta(\alpha_s) = -\alpha_s^2 (b_0 + b_1 \alpha_s + b_2 \alpha_s^2 + \ldots) , \]

\[ b_0 = \frac{11C_A - 2n_f}{12\pi} \quad b_1 = \frac{17C_A^2 - 5C_A n_f - 3C_F n_f}{24\pi^2} = \frac{153 - 19n_f}{24\pi^2} \]

1-Loop $\beta$ function coefficient

2-Loop $\beta$ function coefficient

Asymptotic freedom in the ultraviolet

Confinement (IR slavery?) in the infrared
Asymptotic Freedom

QED:
Vacuum polarization → Charge screening

QCD:
Quark Loops → Also charge screening

But only dominant if > 16 flavors!
Asymptotic Freedom

**QED:**
Vacuum polarization
→ Charge screening

**QCD:**
Gluon Loops
Dominate if \( \leq 16 \) flavors

\[
b_0 = \frac{11C_A - 2n_f}{12\pi}
\]

Spin-1 → Opposite Sign
UV and IR

At low scales

Coupling $\alpha_s(Q)$ actually runs rather fast with $Q$

Perturbative solution diverges at a scale $\Lambda_{QCD}$ somewhere below $\approx 1$ GeV

So, to specify the strength of the strong force, we usually give the value of $\alpha_s$ at a unique reference scale that everyone agrees on: $M_Z$

Full symbols are results based on N3LO QCD, open circles are based on NNLO, open triangles and squares on NLO QCD. The cross-filled square is based on lattice QCD.
QCD has one fundamental parameter

$$\alpha_s(m_Z)_{\overline{MS}}$$

$$\alpha_s(Q^2) = \alpha_s(m_Z^2) \frac{1}{1 + b_0 \alpha_s(m_Z) \ln \frac{Q^2}{m_Z^2} + O(\alpha_s^2)}$$

... and its sibling

$$\Lambda_{QCD}^{(n_f)_{\overline{MS}}}$$

$$\alpha_s(Q^2) = \frac{1}{b_0 \ln \frac{Q^2}{\Lambda^2}}$$

... And all its cousins

$$\Lambda^{(3)} \Lambda^{(4)} \Lambda^{(5)} \Lambda_{CMW} \Lambda_{FSR} \Lambda_{ISR} \Lambda_{MPI}, \ldots$$

... + $n_f$ and quark masses

Will return to these in lecture on Monte Carlos and parton showers
Changing the scale(s)

Why scale variation ~ uncertainty?

Scale dependence of calculated orders must be canceled by contribution from uncalculated ones (+ non-pert)

\[
\alpha_s(Q^2) = \alpha_s(m_Z^2) \frac{1}{1 + b_0 \alpha_s(m_Z) \ln \frac{Q^2}{m_Z^2} + O(\alpha_s^2)}
\]

\[
= \alpha_s(m_Z^2) \left( 1 - b_0 \alpha_s(m_Z) \ln \frac{Q^2}{m_Z^2} + O(\alpha_s^2) \right)
\]

\[
\rightarrow (\alpha_s(Q'^2) - \alpha_s(Q^2)) |M|^2 = \alpha_s^2(Q^2) |M|^2 + \ldots
\]

→ Generates terms of higher order, but proportional to what you already have (\(|M|^2\)) \rightarrow a first naive* way to estimate uncertainty

*warning: some theorists believe it is the only way ... but be agnostic! There are other things than scale dependence ...
"What this year's Laureates discovered was something that, at first sight, seemed completely contradictory. The interpretation of their mathematical result was that the closer the quarks are to each other, the \textit{weaker} is the 'colour charge'. When the quarks are really close to each other, the force is so weak that they behave almost as free particles. This phenomenon is called ‘asymptotic freedom’. The converse is true when the quarks move apart: the force becomes stronger when the distance increases."

\[ \frac{1}{r} \]

\[ \alpha_s(r) \]

\[ 1/r \]

\[ \text{potential} \]

\[ \text{force} \]

\[ \text{charge} \]

\[ \frac{1}{r^2} \]

\[ \frac{1}{r} \]

\[ \infty \]

\[ \alpha_s \]

\[ 0 \]

\[ r \to 0 \]

\[ \text{(Coulomb potential), just less slowly} \]

\[ r \to \infty \]

\[ \text{potential grows linearly as } r \to \infty, \text{ so the force actually becomes constant} \]

\[ \text{(even this is only true in “quenched” QCD. In real QCD, the force eventually vanishes for } r \gg 1\text{fm)} \]