Effects of a **Time-Varying String Tension** & **String Repulsion** in Momentum Space

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**Tau-dependent string tension**
- Physics motivations?
- A primitive model

**Results:** Strangeness-\(p_T\) correlations

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**String repulsion in momentum space**
- Why momentum space?
- Two long, straight, parallel strings

**Work in progress:** towards more complicated topologies

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*with N. Hunt-Smith*
*Publication in preparation*

*with C. Duncan*
*arXiv:1912.09639*

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Peter Skands (Monash University)
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Tension and the Lund String Model

Cornell potential

Potential $V(r)$ between static (lattice) and/or steady-state (hadron spectroscopy) colour-anticolour charges:

$$V(r) = -\frac{a}{r} + \kappa r$$

Lund model built on the asymptotic large-$r$ linear behaviour

But intrinsically only a statement about the late-time / long-distance / steady-state situation. Deviations at early times?

Coulomb effects in the grey area between shower and hadronization? Low-$r$ slope $> \kappa$ favours “early” production of quark-antiquark pairs? + Pre-steady-state effects from a (rapidly) expanding string?
In a recent paper (JHEP 04(2018)145), Berges, Floerchinger, and Venugopalan developed a framework for computing the entanglement between spatial regions for Gaussian states in quantum field theory.

Which they applied to explore an expanding light cone geometry in the [...] Schwinger model for QED in 1+1 space-time dimensions.

- Entanglement entropy is extensive in rapidity at early times.
- “a thermal density matrix for excitations around a coherent field with a time dependent temperature”: $T \propto 1/\tau$

What does this mean in Lund Model context?
I asked an honours student (N. Hunt-Smith) to take our 4\textsuperscript{th} year quantum information course to see if we could parse the entanglement arguments.

He learned a lot but we still didn’t have a dictionary.

We imagine it means the steady state captured by the lattice gets to have thermal excitations characterised by $T \propto 1/\tau$.

But what does \textit{that} mean?

Additional (virtual) quark-antiquark pairs with thermal distribution, which decay away with time?

Allow some of these to become real $\Rightarrow$ new mechanism for string breaks?

First step \textbf{poor man’s model}: to explore effects of a higher effective energy scale and/or steeper potential well being relevant at early times.
As a minimal modification to the existing string model, we studied the consequences of allowing an effective string tension

\[ \kappa_{\text{eff}}(\tau) = \kappa_0 + \Delta \kappa_{\text{therm}}(\tau) \]

where \( \kappa_0 \sim 1 \text{GeV/fm} \) and \( \Delta \kappa_{\text{therm}}(\tau) \propto 1/(\tau + \tau_0) \) with \( \tau_0 \) a regularisation parameter that keeps the effective string tension finite and physically reflects that the string model itself is anyway not appropriate for very early (perturbative) times.

Some Questions:

To model Coulomb effect, study \( \Delta \kappa \sim \frac{d}{dr}(-1/r) = 1/r^2 \)?

(and does \( 1/r^2 \) really map to \( \Delta \kappa \sim 1/\tau^2 \)?)

To model thermal effect, does \( T \sim 1/\tau \) really map to \( \Delta \kappa \sim 1/\tau \)?

(Nuts & bolts not strongly tied to any particular form)
Calculating Tau

To use our modified \( \kappa(\tau) \), need to know the \( \tau \) value of each vertex

In UserHooks, we have access to the \( \Gamma = \kappa^2 x_+ x_- = \kappa^2 \tau^2 \) hyperbolic coordinate (via StringEnd)

\[
\tau = \frac{1}{2} \left( \frac{\sqrt{\Gamma}}{\kappa_0} - k < \tau > - \frac{\Delta \kappa_{\text{max}} k < \tau >}{\kappa_0} \right) + \frac{1}{2} \sqrt{\frac{\Gamma}{\kappa_0^2} - \frac{2 \Delta \kappa_{\text{max}} \sqrt{\Gamma} k < \tau >}{\kappa_0^2} + \frac{\Delta \kappa_{\text{max}}^2 k^2 < \tau >^2}{\kappa_0^2} + \frac{2 \Delta \kappa_{\text{max}} k^2 < \tau >^2}{\kappa_0}}
\]

Solve for \( \tau \) but now using a non-linear relationship (with \( <\tau>=1.2 \text{ GeV}^{-1} \))

\[
\Gamma = \left( \kappa_0 + \Delta \kappa_{\text{max}} \frac{k < \tau >}{\tau + k < \tau >} \right)^2 \tau^2
\]

with \( \Delta \kappa_{\text{max}} \) and \( k \) as free parameters governing the shape of \( \kappa(\tau) \).

(Solution is rather unattractive though.)
Want to generate string breaks with modifiable strangeness ratios and $p_T$ broadening values.

Problem: no easy way to modify the trial probabilities; doChangeFragPar() appears to require constant reinitialisation (and changes are not re-set after use).

Solution for strangeness enhancement: no change of trial probabilities; implement instead as up/down suppression using doVetoFragmentation().

Generate trial breakups as usual, using nominal $P_{s:ud}$

Always accept a strange quark

Accept u,d with probability $P_{accept,ud}(\tau) = (P_{s:ud})^{1-k_0/k(\tau)}$

- In limit $k \gg k_0$: same probability to accept ud as was already generated for s
- In limit $k \sim k_0$: probability to accept ud $\rightarrow 1$ $\Rightarrow$ effective $P_{s:ud}$ unchanged
Transverse Momentum Broadening

Want to generate higher effective $p_T$ broadening values

Again we have the problem that we could not see how to change the trial generation parameters without constant reinitialisation, and such changes do not appear to be re-set after use.

Use the same strategy as for strangeness? (I.e. veto low-$p_T$ hadrons as equivalent to enhancing high-$p_T$ ones)?

StringEnd provides pxHad,pyHad. But bad idea. Using a narrow Gaussian to sample a wider one very quickly becomes extremely inefficient.

Instead: use doChangeFragPar

Re-initialise with a larger StringPT:sigma value + implemented additional method to reset our modifications afterwards.

(Seems overkill / inefficient. To discuss?)
Some Results

Note: this is without retuning to same $\langle N_{ch} \rangle$, $\langle p_T \rangle$, or $\langle$strangeness$\rangle$. Work to be done.
Regardless of technical implementation

Changes to the effective tension ($\tau$ dependence, thermal excitations, or fluctuating string tension - Bialas 99) ➤ mechanism to correlate strangeness and $<p_T>$ without collective effects.

May affect interpretation of data for collective models too?

In perturbative stage, we are generating ss pairs (and others) which do not have a Gaussian $p_T$ spectrum. Then we stop the shower and everything after that is Schwinger. Reasonable (?) that there should be some sort of intermediate/interpolating behaviour?

In general, when looking at departures from Gaussian, the mass and $p_T$ dependence no longer factorises.

What masses to use? Conventional constituent masses probably a good starting point, but much too large for pions?
Consider a pp collision with a single soft gluon exchange

➤ Two parallel straight strings. Idealised picture:

If \( d \ll r_{\text{string}} \) and/or in a Type I SC analogy:

Model as a single (coherent) string, with an initial tension \( \kappa_8 = 2.25 \kappa_3 \) (assuming Casimir scaling) ➤ **Rope Model** (no shoving)

If \( d \gg r_{\text{string}} \) and/or in a Type II SC analogy:

Model as separate strings, with interaction energy proportional to \( 1/d \).

**Shoving model** *(my understanding)*: starting from initial \( d \), do explicit time steps for space-time evolution with repulsive* force (currently modelled as a number of gluons each carrying a small amount of \( p_T \))

*Repulsive: assumes CR modeling effectively accounts for attractive configurations, at least to a first approximation. We shall make the same ansatz.
In Pythia, MPI model is based on perturbative scattering matrix elements (with $p_{T0}$ screening regulator of couplings and propagators)

Strictly speaking, in- and outgoing states are plane waves.

Well-defined momenta ➤ completely delocalised in space:

\[ d \sim O(\Lambda_{QCD}) \]

What does $d$ mean?

Can’t puff and have meal in the mouth …

Fortunately, the momentum is not infinitely resolved. In a calculation with a factorisation scale $Q_F$ the momentum is only defined up to $\Delta Q = O(Q_F)$.

Shower cutoff $Q_{HAD}$ ➤ outgoing shower states localised within $O(1/Q_{HAD})$.

Distances $d > 1/Q_{HAD}$ are meaningful. Distances $d < 1/Q_{HAD}$ not meaningful.
What is \( d \)? (Or at least \(<d>\), to start with)?

Considering only pp: related to \( r_{\text{proton}} \) convoluted with mass distribution
In pp, \( 1/<d> \) is somewhat smaller than \( 1/r_{\text{proton}} \), somewhere in \([\Lambda_{\text{QCD}}, 1 \text{ GeV}]\)

What is \( Q_{\text{HAD}} \)?

Nominally IR cutoff of shower \( \sim 1 \text{ GeV} \): same order of magnitude as \( 1/<d> \)
Another relevant quantity is \( \sqrt{\kappa/\pi} \sim O(\Lambda_{\text{QCD}}) \)

What is \( r_{\text{string}} \)?

A fraction of \( r_{\text{proton}} \), \( r^2 \propto 1/\kappa \) ➤ same order of magnitude as the other numbers
(PS: are we talking about coherence length or penetration depth? I don’t know.)

Option 1: careful modelling dependent on relative \( O(1) \) sizes

Option 2: everything \( O(\Lambda_{\text{QCD}}) \) ➤ put all of it in the same (smeared-out) point
Dynamics determined by time evol. of dofs \( \gg \Lambda_{\text{QCD}} \) (\( p_z \) & perturbative \( p_T \) values)
➤ Stay in momentum space ➤ Simpler modeling. (Some caveats here, ignored.)
Massless quark-antiquark string with invariant mass $W$:

Invariant measure of string length $\sim$ multiplicity of hadrons (with mass $m_0$)

$$\propto \Delta y(m_0) = \ln\left(\frac{W^2}{m_0^2}\right)$$

Note: we take $m_0 \sim m_\rho \sim 0.77 \text{ GeV} \sim 2 \times m_{\text{constituent-quark}}$. Regulates rapidity-span calculation so that we get $\sim$ same results for massless endpoints as when using PYTHIA's constituent-quark masses.

(Assumes all of the invariant mass is available for particle production)

If another string is nearby: assume some of the initial endpoint energy is converted to transverse motion instead

- some fraction of the energy is not available for particle production
- Two-step model. “Compression” (reduce $W^2$) + “Repulsion” (add $p_T^2$)

Idea: preserve string “transverse mass” $W_\perp^2 = W^2 + p_\perp^2 = W_+W_-”$
1. Identical Parallel Strings

Momentum space ➤ assume total effect of repulsion is proportional to rapidity overlap $\Delta y_{ov}$ ($= \Delta y_{string}$ for identical strings)

In principle, could incorporate (physically consistent) knowledge about $d$ via a "form factor"? with $F \to 0$ for $d \to \infty$ and $F \to 1$ for $d \to 0$.

Would probably need to be $F(y,d)$ for more general configurations.

For now, we "hide" $<F>$ in a constant of proportionality.

**Repulsion $p_T$ (total):**

$$p_{\perp R} = \pm c_R \cdot \Delta y_{ov}$$

**Compression:**

$$W^2 \rightarrow W'^2 = \left(1 - \frac{p_{\perp R}^2}{W^2}\right) W^2 \leq W^2$$

Right-moving (massless) endpoint scaled by:

$$W_+ \rightarrow W'_+ = f_+ W_+$$

Left-moving (massless) endpoint scaled by:

$$W_- \rightarrow W'_- = f_- W_-$$

with $f_+ f_- = 1 - \frac{p_{\perp R}^2}{W^2}$ and $f_+ = f_- = f$ for now (by symmetry, for identical strings)
A particularly simple way of representing the repulsion effect would be to boost the $W'$ system by a factor $\beta_T = \frac{p_{TR}}{W'}$

Happy that we had found a very simple way to do the whole thing. But …

Would strings do that?

Transverse boost:

Creates two (forward) jets.
Hadrons at large rapidities get more of the $p_T$
Hadrons at mid-rapidities get no additional $p_T$

What we want: a longitudinally boost-invariant uniform push
Add repulsion $p_T$ as we fragment off the individual hadrons

How much $p_T$ to give to each hadron?

Should be proportional to the (overlapping portion of the) **rapidity span** taken by that hadron

$$\Delta y_i = \ln \left( \frac{W_{i-1}^2}{m_0^2} \right) - \ln \left( \frac{W_i}{m_0^2} \right) = \ln \left( \frac{W_{i-1}^2}{W_i^2} \right)$$

- Rapidity span before hadron $i$ was fragmented off
- Rapidity span after hadron $i$ was fragmented off
- Rapidity span of hadron $i$ independent of $m_0$ parameter

◨ $p_T$ from repulsion given to hadron $i$:

$$p_{\perp,i} = c_R \Delta y_i f_{ov,i} = p_{\perp R} \frac{\Delta y_i f_{ov,i}}{\Delta y_{\text{string}}}$$

with \[\sum_i f_{ov,i} = \frac{\Delta y_{ov}}{\Delta y_{\text{string}}}\] to account for if we step into / out of a region of string overlap.
Parallel Identical Strings: Results

\[ p_{+1} = p_{+2} = 400 \left( 1, 0, \bar{\eta}_\perp \right) \text{ GeV} \]
\[ p_{-1} = p_{-2} = 400 \left( 0, 1, \bar{\eta}_\perp \right) \text{ GeV} \]

Default (random) fragmentation \( p_T \) + repulsion \( p_T \)

Lower <\( p_T \)> for shoving model: soft gluons increase multiplicity faster than total \( p_T \)?

Repulsion component only
(\text{obtained by setting StringPT:sigma=0})

Average primary hadron \( p_\perp \) vs string length taken for \( c_R = 0.2 \text{ GeV} \)

\( <p_T> \) vs string length taken

\( \Delta y \)

Amount of string length taken

Duncan & PS, arXiv:1912.09639
Average primary hadron $p_\perp$ vs hadron rapidity for $c_R = 0.2$ GeV

- **Overlap Region**
- **Lund Model (Baseline)**
- **Lund + Repulsion**
- **Repulsion Component**
- **Shoving Model**

**Uniform: this is what we wanted**

Duncan & PS, arXiv:1912.09639

Varying the strength of $c_R$

Duncan & PS, arXiv:1912.09639
(Effect of Hadron Decays)

Figure 11: Illustration of the net reduction of average hadron $p_T$ caused by allowing excited primary hadrons (solid histograms) to decay (dot-dashed histograms), for the baseline Lund model (red) and our fragmentation repulsion model (blue). The example configuration is the symmetric parallel two-string configuration described in Sec. 3; the primary-hadron spectra are the same as those in Fig. 3.

In Fig. 10, we plot the results for the two-particle cumulant for the symmetric two-string configuration at the level of primary hadrons, as a function of the repulsion constant $c_R$. There are four curves in the plot. The first curve, labelled 'Symmetric' is the simplest two-string configuration, considered in Sec. 3. In this configuration, there is no preferred direction, and it takes larger values of the repulsion constant to overcome the Gaussian transverse momentum distribution of the Lund fragmentation model, and to have a significant effect on the cumulant.

The three other curves are variations on the configuration described in Sec. 5.1 where the two strings each have a boost of $p_T=0$. The variations occur when one adds the repulsion $p_T$ to the primary hadrons during fragmentation. The curves are labelled according to the direction in which the repulsion $p_T$ is added with respect to the given string's overall boost direction. If we add the repulsion $p_T$ in the same direction as the string's motion, we can greatly enhance the two-particle cumulant. If instead we add it in the opposite direction, we at first reduce the two-particle cumulant, but as the repulsion gets larger, the cumulant begins to increase. Lastly, if we add the repulsion $p_T$ perpendicularly to the string's motion we greatly reduce the cumulant, but at large values of the repulsion constant, the rate of decrease begins to level out.

We obtain analogous results for the general configuration in the right panel of Fig. 10, though the cumulant for all values of $c_R$ is less than for the symmetric case, due to the smaller overlap in rapidity.

We compared the symmetric parallel configuration in our fragmentation repulsion framework to the analogous configuration in the shoving model, and found that the two-particle cumulant is significantly smaller for the shoving model, at least with the parameter set described in App. B. For the shoving model, we calculated the two-particle cumulant to be $c_2^{(2)}=0.00957$ (averaged over 200,000 events), which is of the order of the baseline Lund model.
Towards more general topologies

It is rare that nature hands you two identical straight strings

- Asymmetric straight parallel strings
- Strings with a relative boost
- Strings with a relative rotation
- Strings with heavy endpoints
- More than 2 strings
- Strings with gluon kinks
- Junction strings
- Finite-distance effects
Towards more general topologies

It is rare that nature hands you two identical straight strings

Asymmetric straight parallel strings
Strings with a relative boost
Strings with a relative rotation
Strings with heavy endpoints
More than 2 strings
Strings with gluon kinks
Junction strings
Finite-distance effects

These are the cases we managed to consider in Duncan & PS, arXiv:1912.09639
Asymmetric Strings

Computation of rapidity overlap (and hence $p_{TR}$) still straightforward

Main new question is whether to allow $p_z$ exchange: “longitudinal recoil”? 

Regardless of $p_z$ strategy, the rescaling factors must satisfy:

\[
\begin{align*}
    f_+ f_- &= f_1^2 = 1 - \frac{p_{1,R}^2}{W_1^2} \\
    f_+ f_- &\neq f_2^2 = 1 - \frac{p_{1,R}^2}{W_2^2}
\end{align*}
\]

Longitudinal momentum conservation, $\Delta p_{z1} = -\Delta p_{z2}$:

\[
(1 - f_+) W_{+1} - (1 - f_-) W_{-1} = (1 - f_-) W_{-2} - (1 - f_+) W_{+2}
\]

Need one more constraint. For now, we impose no $p_z$ exchange (for simplicity; not convinced it is consistent with Lorentz invariance: $p_z$ frame dependent. Reasonable starting point(?): no $\Delta p_z$ in frame with centre of overlap at $y=0$).
Asymmetric Strings: Solutions

Assuming no $p_z$ exchange:

\[ W'_{-i} = f_{-i} W_{-i} = \sqrt{W_{Li}^2 + W_i^2 f_i^2 - W_{Li}} , \]

\[ W'_{+i} = f_{+i} W_{+i} = \sqrt{W_{Li}^2 + W_i^2 f_i^2 + W_{Li}} . \]

(reproduces the symmetric case in the limit $W_{+i} = W_{-i}$ i.e. $W_{Li} = 0$)

By construction longitudinal momentum is conserved:

\[ W'_{+i} - W'_{-i} = W_{+i} - W_{-i} . \]

Energy, however, is reduced (compression):

\[ E'_{i} = \frac{W'_{+i} + W'_{-i}}{2} = E_i \sqrt{1 - \frac{p_{\perp,R}^2}{E_i^2}} \]

We regain the “lost” energy by giving the primary hadrons the repulsion $p_{\perp}$ and putting them on-shell again, with the string remnant absorbing the remaining energy.
Asymmetric parallel strings: Results

\[ p_{+1} = 1200 \begin{pmatrix} 1, 0, 0 \end{pmatrix} \text{ GeV}, \]
\[ p_{-1} = 300 \begin{pmatrix} 0, 1, 0 \end{pmatrix} \text{ GeV}, \]
\[ p_{+2} = 100 \begin{pmatrix} 1, 0, 0 \end{pmatrix} \text{ GeV}, \]
\[ p_{-2} = 1000 \begin{pmatrix} 0, 1, 0 \end{pmatrix} \text{ GeV}, \]

Although we used pretty long strings (we thought), effects of partial overlaps still somewhat obscured by endpoint falloffs.
Topologies with a relative transverse boost

2. Rescale string ends similarly to before. This causes the ends to lose some \( p_T \). Added to \( p_T \) reservoir to be added back during fragmentation.

Alternative: boost compressed strings so they regain their original \( p_T \)? Reduce \( p_z \), then \( E \) to bring back on shell?

\[
\begin{align*}
p_1 &= E(1, \sin \theta, 0, -\cos \theta) \\
p_2 &= E(1, \sin \theta, 0, \cos \theta) \\
p_3 &= E(1, -\sin \theta, 0, -\cos \theta) \\
p_4 &= E(1, -\sin \theta, 0, \cos \theta)
\end{align*}
\]

Boost \( \beta = \pm \sin(\theta) = 0.1 \)

1. Evaluate rapidity overlap along common axis (smaller than the individual string CM rapidity spans) ➤ total \( p_{TR} \)

3. Hadron rapidity spans projected onto common axis:

\[
\Delta y_{\text{eff}} = \frac{\Delta y_{\text{string}}}{\Delta y^*_{\text{string}}} \frac{\Delta y^*_{\text{taken}}}{\Delta y^*_{\text{ov}}}
\]

In reality, soft hadrons should have \( f_{ov} \sim 1 \)?
**Results: Boosted topologies**

**Symmetric**

Average primary hadron $p_\perp$ vs hadron rapidity for $c_R = 0.4$ GeV

- Lund Model (Baseline)
- Lund Model (Long.)
- Lund + Repulsion
- Repulsion Component

**Asymmetric**

(same as the one used earlier with boost $\beta=0.1$ in opposite directions)

Average primary hadron $p_\perp$ vs hadron rapidity for $c_R = 0.4$ GeV

- Lund Model (Baseline)
- Lund Model (Long.)
- Lund + Repulsion
- Repulsion Component

Subtlety: which direction? We assume same direction as relative boost, with random component added to have well-defined behaviour in boost→0 limit.
Two-Particle Cumulants

To connect with collective-flow / HI observables, we considered the two-particle cumulant

\[ c_2 \{2\} = \left\langle e^{2i(\phi_i - \phi_j)} \right\rangle = \frac{2}{n(n-1)} \sum_{i<j} \cos(2(\phi_i - \phi_j)) \]

Two-particle cumulant \(c_2\{2\}\) of primary hadrons as a function of \(c_R\)

- Symmetric
- Symmetric, boosted (+)
- Symmetric, boosted (-)
- Symmetric, boosted (⊥)

Primary Hadrons

With hadron decays: smaller magnitude but same trends

Two-particle cumulant \(c_2\{2\}\) of final-state hadrons as a function of \(c_R\)

- Symmetric
- Symmetric, boosted (+)
- Symmetric, boosted (-)
- Symmetric, boosted (⊥)
Much theoretical activity to understand, model, and disentangle signs of collective effects in pp collisions.

Interesting to take a step further back: re-examine the modelling of the fragmentation of a single string.

- Grey zone between shower, $V_{\text{Coulomb}}$, and asymptotic string descriptions.
- Expanding geometry $\leftrightarrow$ entanglement $\leftrightarrow$ effective thermal effects?
- E.g., a **$\tau$-dependent effective string tension** can generate a $<p_T>$ vs strangeness correlation. (Fluctuating string tension likewise?)
  
  I have no good LEP measurements on $<p_T>$ vs strangeness? Only inclusive $<p_{T\text{in}}>$, $<p_{T\text{out}}>$ and (limited) PID x spectra dominated by $p_z$.

First steps towards a simple framework for momentum-space modelling of string-string repulsion effects

- Basic framework: 2-step "compression" + "fragmentation repulsion"
  
  So far considered only rather simple / textbook sort of setups. Interested to discuss merits (or showstoppers) to motivate further work.
Shoving Model Parameters

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Table 1: Input parameters used in Fig. 3 for the shoving model.
(Note on fluctuating string tension)

Following a suggestion by Bialas (hep-ph/9909417), a recent study (Pirner, Kopeliovich, Reygers, arXiv:1810.0473) allowed for a fluctuating $\kappa$.

Flux tube size $r^2 \propto 1/\kappa$. Allow Gaussian fluctuations with $\kappa^2 = \lambda$ and

$$P(\lambda) d\lambda = \sqrt{\frac{2}{\pi \mu}} e^{-\frac{\lambda^2}{2\mu}} d\lambda.$$

with $\langle \kappa \rangle \equiv \langle \lambda^2 \rangle = \mu = \int_0^\infty \lambda^2 P(\lambda) d\lambda$.

Extremely simplified pion spectrum:

$$\frac{dN}{d^2 p_\perp} = N_0 e^{-\sqrt{\frac{2\pi (m^2_q + p^2_\perp/2)}{\langle \kappa \rangle}}}$$

They fit $<\kappa>$ from $dN/dp_T$ in $[0.5, 1.4]$ GeV in 4 multiplicity classes (using a Tsallis function to extrapolate for the total Nch to $p_T=0$)

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<th>$\frac{dN_{ch}}{d\eta}_{\eta=0}$</th>
<th>$\langle \kappa \rangle$ in GeV$^2$</th>
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Crude techniques but the idea of extracting an effective average tension from $<p_T>(N_{ch})$ and relating that to strangeness enhancement may have merit.