What we actually see when we look at a “jet” (or inside a proton)

An ever-repeating self-similar pattern of quantum fluctuations

At increasingly smaller energies or distances: scaling \( (\text{modulo } \alpha_s(Q) \text{ scaling violation}) \)

To our best knowledge, this is what a fundamental (‘elementary’) particle really looks like
What we actually see when we look at a “jet” (or inside a proton)

An ever-repeating self-similar pattern of quantum fluctuations

At increasingly smaller energies or distances: **scaling** (modulo $\alpha_s(Q)$ scaling violation)

To our best knowledge, this is what a fundamental (‘elementary’) particle really looks like

Nature makes copious use of such structures - **Fractals**
Most bremsstrahlung is driven by **divergent propagators** → simple structure

**Amplitudes factorise in singular limits** (→ universal “scale-invariant” or “conformal” structure)

$$|\mathcal{M}_{F+1}(\ldots, a, b, \ldots)|^2 \xrightarrow{a \parallel b} g_s^2 \mathcal{C} \frac{P(z)}{2(p_a \cdot p_b)} |\mathcal{M}_F(\ldots, a + b, \ldots)|^2$$

**Gluon j → “soft”:**

$$|\mathcal{M}_{F+1}(\ldots, i, j, k, \ldots)|^2 \xrightarrow{j \rightarrow 0} g_s^2 \mathcal{C} \frac{(p_i \cdot p_k)}{(p_i \cdot p_j)(p_j \cdot p_k)} |\mathcal{M}_F(\ldots, i, k, \ldots)|^2$$

+ **scaling violation:** $g_s^2 \rightarrow 4\pi\alpha_s(Q^2)$

**Can apply this many times** → **nested factorizations**
Naively, QCD radiation suppressed by $\alpha_s \approx 0.1$

$\Rightarrow$ Truncate at fixed order = LO, NLO, …

But beware the jet-within-a-jet-within-a-jet …

**Example:** 100 GeV can be “soft” at the LHC

SUSY pair production at LHC$_{14}$, with $M_{\text{SUSY}} \approx 600$ GeV

| FIXED ORDER pQCD | $\sigma_{\text{tot}}$ [pb] $\tilde{g}\tilde{g}$ $\tilde{u}_L\tilde{g}$ $\tilde{u}_L\tilde{u}_L^*$ $\tilde{u}_L\tilde{u}_L$ $TT$ |
|------------------|-----------------|-------------------|-----------------|-----------------|-----------------|
| $p_T,j > 100$ GeV | $\sigma_{0j}$ | 4.83 | 5.65 | 0.286 | 0.502 | 1.30 |
| inclusive X + 1 “jet” | $\sigma_{1j}$ | 2.89 | 2.74 | 0.136 | 0.145 | 0.73 |
| inclusive X + 2 “jets” | $\sigma_{2j}$ | 1.09 | 0.85 | 0.049 | 0.039 | 0.26 |

| $p_T,j > 50$ GeV | $\sigma_{0j}$ | 4.83 | 5.65 | 0.286 | 0.502 | 1.30 |
| $\sigma_{1j}$ | 5.90 | 5.37 | 0.283 | 0.285 | 1.50 |
| $\sigma_{2j}$ | 4.17 | 3.18 | 0.179 | 0.117 | 1.21 |

(Computed with SUSY-MadGraph)

All the scales are high, $Q >> 1$ GeV, so perturbation theory **should** be OK
Why are Fixed-Order QCD matrix elements not enough?

F.O. QCD requires **Large scales** ($\alpha_s$ small enough to be perturbative $\rightarrow$ high-scale processes)

F.O. QCD also requires **No hierarchies**

Bremsstrahlung poles $\propto 1/Q^2$ integrated over phase space $\propto dQ^2 \rightarrow$ logarithms $\rightarrow$ large if upper and lower integration limits are hierarchically different
So it’s not like you can put a cut at $X$ (e.g., 50, or even 100) GeV and say: “ok, now fixed-order matrix elements will be OK”

The hard scale $Q_{\text{HARD}}$ of your process will “start off” the fractal

Sooner or later you will resolve bremsstrahlung structure (when $Q_{\text{Resolved}}/Q_{\text{HARD}} < 1$)

Extra radiation:

Will generate corrections to your kinematics

Is an unavoidable aspect of the quantum description of quarks and gluons (no such thing as a “bare” quark or gluon; they always depend on how you look at them)

Extra jets from bremsstrahlung can be important combinatorial background especially if you are looking for decay jets of similar $p_T$ scales (often, $\Delta M \ll M$)

This is what parton showers are for
For any basic process

\[ d\sigma_X = \begin{array}{c} \checkmark \end{array} \]  
(calculated process by process)

\[
d\sigma_{X+1} \sim NCg_s^2 \frac{ds_{i1}}{s_{i1}} \frac{ds_{1j}}{s_{1j}} d\sigma_X \begin{array}{c} \checkmark \end{array}
\]

\[
d\sigma_{X+2} \sim NCg_s^2 \frac{ds_{i2}}{s_{i2}} \frac{ds_{2j}}{s_{2j}} d\sigma_{X+1} \begin{array}{c} \checkmark \end{array}
\]

\[
d\sigma_{X+3} \sim NCg_s^2 \frac{ds_{i3}}{s_{i3}} \frac{ds_{3j}}{s_{3j}} d\sigma_{X+2} \ldots
\]

NB: here just iterating a single eikonal emission; should really sum over all emitters.
Could also have built an approximation from iterating collinear emissions (DGLAP)
For any basic process $d\sigma_X = \sqrt{\text{calculated process by process}}$

$$d\sigma_{X+1} \sim NC^2 g_s^2 \frac{ds_{i1}}{s_{i1}} \frac{ds_{1j}}{s_{1j}} d\sigma_X \quad \checkmark$$

$$d\sigma_{X+2} \sim NC^2 g_s^2 \frac{ds_{i2}}{s_{i2}} \frac{ds_{2j}}{s_{2j}} d\sigma_{X+1} \quad \checkmark$$

$$d\sigma_{X+3} \sim NC^2 g_s^2 \frac{ds_{i3}}{s_{i3}} \frac{ds_{3j}}{s_{3j}} d\sigma_{X+2} \quad \ldots$$

**Singularities:** universal (mandated by gauge theory)

**Non-singular terms:** process-dependent

$$\frac{|\mathcal{M}(Z^0 \rightarrow q_i g_j \bar{q}_k)|^2}{|\mathcal{M}(Z^0 \rightarrow q_i \bar{q}_K)|^2} = g_s^2 2C_F \left[ \frac{2s_{ik}}{s_{ij} s_{jk}} + \frac{1}{s_{IK}} \left( \frac{s_{ij}}{s_{jk}} + \frac{s_{jk}}{s_{ij}} \right) \right]$$

```
"SOFT"
```

$$\frac{|\mathcal{M}(H^0 \rightarrow q_i g_j \bar{q}_k)|^2}{|\mathcal{M}(H^0 \rightarrow q_i \bar{q}_K)|^2} = g_s^2 2C_F \left[ \frac{2s_{ik}}{s_{ij} s_{jk}} + \frac{1}{s_{IK}} \left( \frac{s_{ij}}{s_{jk}} + \frac{s_{jk}}{s_{ij}} + 2 \right) \right]$$

```
"SOFT"
```

```
"COLLINEAR"
```

NB: here just iterating a single eikonal emission; should really sum over all emitters.

Could also have built an approximation from iterating collinear emissions (DGLAP)
For any basic process \( d\sigma_X = \checkmark \) (calculated process by process)

\[
\begin{align*}
    d\sigma_{X+1} &\sim N C 2 g_s^2 \frac{d s_{i1}}{s_{i1}} \frac{d s_{1j}}{s_{1j}} d\sigma_X \quad \checkmark \\
    d\sigma_{X+2} &\sim N C 2 g_s^2 \frac{d s_{i2}}{s_{i2}} \frac{d s_{2j}}{s_{2j}} d\sigma_{X+1} \quad \checkmark \\
    d\sigma_{X+3} &\sim N C 2 g_s^2 \frac{d s_{i3}}{s_{i3}} \frac{d s_{3j}}{s_{3j}} d\sigma_{X+2} \ldots
\end{align*}
\]

**Iterated factorization**

Gives us a universal approximation to \( \infty \)-order tree-level cross sections.

Exact in singular (strongly ordered) limit.

Non-singular terms (non-universal) \( \rightarrow \) Uncertainties for hard radiation

But something is not right ... Total \( \sigma \) would be infinite ...
Coefficients of the Perturbative Series

**Dots and Legs**

The corrections from Quantum Loops are **missing**

Universality (scaling)

Jet-within-a-jet-within-a-jet-...
RECAP: ADDING JETS AT FIXED ORDER

Born @ LO

\[ \sigma_{\text{Born}} = \int \left| M_X^{(0)} \right|^2 \]

Born + n @ LO

\[ \sigma_{X+1}^{LO}(R) = \int_R \left| M_{X+1}^{(0)} \right|^2 \]

\[ \frac{|M_{X+1}|^2}{|M_X|^2} \propto g_s^2 2C_F \left[ \frac{2s_{ik}}{s_{ij}s_{jk}} + \frac{1}{sIK} \left( \frac{s_{ij}}{s_{jk}} + \frac{s_{jk}}{s_{ij}} \right) \right] \]

Divergent (when \( s_{ij} \) and/or \( s_{jk} \) \( \to \) 0): Integral \( \to \) Logarithms

\[ \Rightarrow R = \text{some "Infrared Safe" phase space region (E.g., cut on } p_{\perp}, \Delta R) \]

Careful not to take it too low!
UNITARITY (AT NLO)

NLO:

\[
\sigma_X^{\text{NLO}} = \int |M_X^{(0)}|^2 + \int |M_{X+1}^{(0)}|^2 + \int 2\text{Re}[M_X^{(1)} M_X^{(0)*}] 
\]

IR singularities
(from poles of propagators going on shell when integrating to \(Q^2 \to 0\))

IR singularities
(from poles of propagators going on shell when integrating over gluon virtuality)

In IR limits, the \(X+1\) final state is indistinguishable from an \(X+0\) one
\(\rightarrow\) singularities must always* sum together (& they cancel!)

Example:

\[
\sigma_{\text{NLO}}(e^+e^- \to q\bar{q}) = \sigma_{\text{LO}}(e^+e^- \to q\bar{q}) \left( 1 + \frac{\alpha_s(E_{\text{CM}})}{\pi} \right) + \mathcal{O}(\alpha_s^2) 
\]

Sum of real and virtual \(O(\alpha_s)\) nonsingular; no IR regulator dependence

*) for so-called IR safe observables; discussed in Lecture 3
Probability for nothing to happen (\sim virtual + unresolved-real) + Probability for something to happen (\sim resolved real) = 1

Unitarity: \text{sum(probability)} = 1

**Kinoshita-Lee-Nauenberg**

(sum over degenerate quantum states = finite; infinities must cancel)

\[
\text{Loop} = - \int \text{Tree} + F
\]

Parton Showers neglect F \rightarrow "Leading-Logarithmic" (LL) Approximation

**Imposed by Event evolution:** "detailed balance"

When \((X)\) branches to \((X+1)\): **Gain** one \((X+1)\). **Loose** one \((X)\).

Differential equation with evolution kernel

\[
\frac{d\sigma x + 1}{d\sigma x}
\]

(or, typically, a soft/collinear approximation thereof)

Evolve in some measure of \textit{resolution} \sim hardness, 1/time ... \sim fractal scale

+ account for scaling violation via quark masses and \(g_s^2 \rightarrow 4\pi\alpha_s(Q^2)\)

\rightarrow includes both real (tree) and virtual (loop) corrections, to arbitrary order
(E.g., starting from QCD $2 \rightarrow 2$)

\[ Q \sim Q_{\text{HARD}} \quad \quad \quad \quad Q_{\text{HARD}}/Q < \text{“A few”} \]

At most inclusive level
“Everything is 2 jets”

At (slightly) finer resolutions, some events have 3, or 4 jets

At high resolution, most events have $>2$ jets

Fixed order:
\[ \sigma_{\text{inclusive}} \]

Fixed order:
\[ \sigma_{X+n} \sim \alpha_s^n \sigma_X \]

Fixed order diverges:
\[ \sigma_{X+n} \sim \alpha_s^n \ln^{2n}(Q/Q_{\text{HARD}}) \sigma_X \]

Scale Hierarchy!

Unitarity: Reinterpret as number of emissions diverging, while cross section remains $\sigma_{\text{inclusive}}$.
What we need is a **differential equation**

Boundary condition: a few partons defined at a high scale \((Q_F)\)

Then evolves (or “runs”) that parton system down to a low scale (the hadronization cutoff \(\sim 1 \text{ GeV}\) \(\rightarrow\) It’s an evolution equation in \(Q_F\)

**Close analogue: nuclear decay**

Evolve an unstable nucleus. Check if it decays + follow chains of decays.

- **Decay constant**
  \[
  \frac{dP(t)}{dt} = c_N
  \]

- **Probability to remain undecayed in the time interval \([t_1,t_2]\)**
  \[
  \Delta(t_1,t_2) = \exp \left( - \int_{t_1}^{t_2} c_N \, dt \right) = \exp \left( -c_N \, \Delta t \right) = 1 - c_N \Delta t + O(c_N^2)
  \]
  \(\Delta(t_1,t_2)\) : “Sudakov Factor”

- **Decay probability per unit time**
  \[
  \frac{dP_{\text{res}}(t)}{dt} = \frac{-d\Delta}{dt} = c_N \, \Delta(t_1,t)
  \]
  (respects that each of the original nuclei can only decay if not decayed already)
THE SUDAKOV FACTOR

In nuclear decay, the Sudakov factor counts:

How many nuclei remain undecayed after a time $t$

Probability to remain undecayed in the time interval $[t_1,t_2]$

$$\Delta(t_1, t_2) = \exp \left( - \int_{t_1}^{t_2} c_N \, dt \right) = \exp (-c_N \Delta t)$$

The Sudakov factor for a parton system "counts":

The probability that the parton system doesn’t evolve (branch) when we run the factorization scale ($\sim 1$/time) from a high to a low scale

(i.e., that there is no state change)

Evolution probability per unit "time"

$$\frac{dP_{\text{res}}(t)}{dt} = -\frac{d\Delta}{dt} = c_N \Delta(t_1, t)$$

(replace $t$ by shower evolution scale)

(replace $c_N$ by proper shower evolution kernels)
NUCLEAR DECAY

\[
\Delta(t_1, t_2) = \exp \left( - \int_{t_1}^{t_2} dt \frac{d\mathcal{P}}{dt} \right)
\]

Nuclei remaining undecayed after time \( t \)
1. For each evolver, generate a random number $R \in [0,1]$

Solve equation $R = \Delta(t_1, t)$ for $t$ (with starting scale $t_1$)

- Analytically for simple splitting kernels,
- else numerically and/or by trial+veto
- $\rightarrow t$ scale for next (trial) branching

2. Generate another Random Number, $R_z \in [0,1]$

To find second (linearly independent) phase-space invariant

Solve equation $R_z = \frac{I_z(z, t)}{I_z(z_{\max}(t), t)}$ for $z$ (at scale $t$)

With the “primitive function” $I_z(z, t) = \int_{z_{\min}(t)}^{z} dz \left( \frac{d\Delta(t')}{dt'} \right)_{t'=t}$

3. Generate a third Random Number, $R_\varphi \in [0,1]$

Solve equation $R_\varphi = \varphi/2\pi$ for $\varphi \rightarrow$ Can now do 3D branching

Accept/Reject based on full kinematics. Update $t_1 = t$. Repeat.
Start from an arbitrary lowest-order process (green = QFT amplitude squared)

Parton showers generate the (LL) bremsstrahlung terms of the rest of the perturbative series (approximate infinite-order resummation)

Note! LL ≠ full QCD! (→ matching, merging, MECs)
WHAT ARE THE EVOLUTION KERNELS?

Recall: two universal (bremsstrahlung) limits:

Collinear (DGLAP) Limit: two partons becoming parallel

Partons $ab \rightarrow “collinear”: P(z) = \text{DGLAP splitting kernels, with } z = \text{energy fraction } = E_a/(E_a+E_b)$

$$|M_{F+1}(\ldots, a, b, \ldots)|^2 \xrightarrow{a||b} \frac{g_s^2 C}{2(p_a \cdot p_b)} \frac{P(z)}{|M_F(\ldots, a + b, \ldots)|^2}$$

Soft (eikonal) Limit: an emitted gluon having vanishing energy

Gluon $j \rightarrow “soft”: Coherence \rightarrow \text{Parton } j \text{ really emitted by } (i,k) \text{ “colour antenna”}$

$$|M_{F+1}(\ldots, i, j, k, \ldots)|^2 \xrightarrow{j \rightarrow 0} \frac{g_s^2 C}{(p_i \cdot p_j)(p_j \cdot p_k)} \frac{(p_i \cdot p_k)}{|M_F(\ldots, i, k, \ldots)|^2}$$

→ can build different types of parton showers (and, in general, different kinds of resummations)
Starting from collinear (parton) limit:

DGLAP evolution, collinear factorisation (MSbar PDFs)

“Conventional Parton Showers” : earliest shower models

Modified for correct soft limits: \textit{angular ordering} * (or vetos), (CS) Dipole showers

\begin{align*}
\text{Parton Shower (DGLAP)} & \quad \text{Coll}(I) \quad a_I \\
\text{Coherent Parton Shower (HERWIG}\ [12,40], \text{PYTHIA6}\ [11]) & \quad \Theta_I a_I \\
\text{Global Dipole-Antenna (ARIADNE}\ [17], \text{GGG}\ [36], \text{WK}\ [32], \text{VINCIA}) & \quad a_{IK} + a_{HI} \\
\text{Sector Dipole-Antenna (LP}\ [41], \text{VINCIA}) & \quad \Theta_{IK} a_{IK} + \Theta_{HI} a_{HI} \\
\text{Partitioned-Dipole Shower (SK}\ [23], \text{NS}\ [42], \text{DTW}\ [24], \text{PYTHIA8}\ [38], \text{SHERPA, DIRE}) & \quad a_{IK} + a_{IK} + a_{HI}
\end{align*}

Starting from soft (dipole) limit:

DLA (only double-pole piece), eikonal approximations

Extended to include DGLAP collinear limits: (Lund) Dipole / Antenna showers

\textit{angular ordering} * coherence only in an averaged sense; discussed later

\begin{align*}
\text{HENWIG} & \quad a_I \\
\text{PYTHIA} & \quad \Theta_I a_I \\
\text{HERWIG CS, SHERPA CS, DIRE} & \quad a_{IK} + \Theta_{HI} a_{HI} \\
\text{ARIADNE} & \quad a_{IK} + a_{IK} + a_{HI}
\end{align*}
DGLAP: from *collinear limit* of MEs \((p_b+p_c)^2 \rightarrow 0\)

+ evolution equation from invariance with respect to \(Q_F \rightarrow \text{RGE}\)

\[
dP_a = \sum_{b,c} \frac{\alpha_{abc}}{2\pi} P_{a \rightarrow bc}(z) \, dt \, dz .
\]

\[
P_{q \rightarrow qg}(z) = C_F \frac{1 + z^2}{1 - z} ,
\]
\[
P_{g \rightarrow gg}(z) = N_C \frac{(1 - z(1 - z))^2}{z(1 - z)} ,
\]
\[
P_{g \rightarrow q\bar{q}}(z) = T_R (z^2 + (1 - z)^2) ,
\]
\[
P_{q \rightarrow q\gamma}(z) = e_q^2 \frac{1 + z^2}{1 - z} ,
\]
\[
P_{\ell \rightarrow \ell\gamma}(z) = e_\ell^2 \frac{1 + z^2}{1 - z} ,
\]

\[
dt = \frac{dQ^2}{Q^2} = d\ln Q^2
\]

---

**EXAMPLE: DGLAP KERNELS**

**NB:** dipoles, antennae, also have DGLAP kernels as their collinear limits
QED: Chudakov effect (mid-fifties)

Illustration by T. Sjöstrand

emulsion plate  reduced ionization  normal ionization
Physics: (applies to any gauge theory)

Interference between emissions from colour-connected partons (e.g. i and k) $\rightarrow$ coherent dipole patterns

(More complicated multipole effects beyond leading colour; ignored here)

DGLAP kernels, though incoherent a priori, can reproduce this pattern (at least in an azimuthally averaged sense) by angular ordering

Start from the M.E. factorisation formula in the soft limit

$$\frac{E_j^2 (p_i \cdot p_k)}{(p_i \cdot p_j)(p_j \cdot p_k)} = \frac{1 - \cos \theta_{ik}}{(1 - \cos \theta_{ij})(1 - \cos \theta_{jk})} = \frac{1 - \cos \theta_{ik}}{(1 - \cos \theta_{ij})(1 - \cos \theta_{jk})} \pm \frac{1}{2(1 - \cos \theta_{ij})} \mp \frac{1}{2(1 - \cos \theta_{jk})}$$

Soft Eikonal Factor

(write out 4-products)

Add and subtract $1/(1-\cos \theta_{ij})$ and $1/(1-\cos \theta_{ik})$ to isolate $ij$ and $jk$ collinear pieces

$$\int_0^{2\pi} \frac{d\varphi_{ij}}{4\pi} \left( \frac{1 - \cos \theta_{ik}}{(1 - \cos \theta_{ij})(1 - \cos \theta_{jk})} + \frac{1}{1 - \cos \theta_{ij}} - \frac{1}{1 - \cos \theta_{jk}} \right) = \frac{1}{2(1 - \cos \theta_{ij})} \left( 1 + \frac{\cos \theta_{ij} - \cos \theta_{ik}}{|\cos \theta_{ij} - \cos \theta_{ik}|} \right)$$

Take the $ij$ piece and integrate over azimuthal angle $d\varphi_{ij}$ (using explicit momentum representations)

$\Rightarrow$ Soft radiation averaged over $\varphi_{ij}$:

$$\rightarrow \frac{1}{1 - \cos \theta_{ij}}$$

if $\theta_{ij} < \theta_{ik}$; otherwise 0

Note: Dipole & antenna showers include this effect point by point in $\varphi$ (without averaging)

kill radiation outside ik opening angle
Example: quark-quark scattering in hadron collisions

Consider, for instance, scattering at 45°

2 possible colour flows:

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Figure 4: Angular distribution of the first gluon emission in \( qq \rightarrow qq \) scattering at 45°, for the two different color flows. The light (red) histogram shows the emission density for the forward flow, and the dark (blue) histogram shows the emission density for the backward flow.

Another nice physics example is the SM contribution to the Tevatron top-quark forward-backward asymmetry from coherent showers, see: PS, Webber, Winter, JHEP 1207 (2012) 151
INITIAL-STATE VS FINAL-STATE EVOLUTION

FSR: $p^2 > 0$

Virtualities are Timelike: $p^2 > 0$
Start at $Q^2 = Q_F^2$
"Forwards evolution"

ISR: $p^2 = t < 0$

Virtualities are Spacelike: $p^2 < 0$
Start at $Q^2 = Q_F^2$
Constrained backwards evolution towards boundary condition = proton

Separation meaningful for collinear radiation, but not for soft ...
A tricky aspect for many parton showers. Illustrates that quantum ≠ classical!

Who emitted that gluon?

Real QFT = sum over amplitudes, then square → interference (IF coherence)
Respected by dipole/antenna languages (and by angular ordering, azimuthally averaged), but not by conventional DGLAP (→ all PDFs are “wrong”)

Separation meaningful for collinear radiation, but not for soft …
The final states generated by a shower algorithm will depend on

1. The choice of perturbative evolution variable(s) \( t^{[i]} \).

2. The choice of phase-space mapping \( \frac{d\Phi_{n+1}^{[i]}}{d\Phi_n} \).

3. The choice of radiation functions \( a_i \), as a function of the phase-space variables.

4. The choice of renormalization scale function \( \mu_R \).

5. Choices of starting and ending scales.

→ gives us additional handles for uncertainty estimates, beyond just \( \mu_R \)
(+ ambiguities can be reduced by including more pQCD → matching!)
Recently, HERWIG, PYTHIA & SHERPA all published papers on automated calculations of shower uncertainties (based on tricks with the Sudakov algorithm)

**Weight of event** = { 1, 0.7, 1.2, ... }
SUMMARY: TWO WAYS TO COMPUTE QUANTUM CORRECTIONS

**Fixed Order Paradigm:** consider a single physical process

Explicit solutions, process-by-process (to some extent automated)
- Standard-Model: typically NLO or NNLO
- Beyond-SM: typically LO or NLO

Accurate for hard process, to given perturbative order
Limited generality

**Event Generators (Showers):** consider all physical processes

Universal solutions, applicable to any/all processes
- Process-dependence = subleading correction (→ matrix-element corrections)

Maximum generality
- Common property of all processes is, e.g., limits in which they factorise!

Accurate in strongly ordered (soft/collinear) limits (=bulk of radiation)
From \textbf{MS} to \textbf{MC}


\[
P(\alpha_s, z) = \frac{\alpha_s}{2\pi} C_F \frac{A^{(1)}}{1 + z^2} + \left( \frac{\alpha_s}{\pi} \right)^2 \frac{A^{(2)}}{1 - z}
\]

Eg Analytic resummation (in Mellin space): General Structure

\[
\propto \exp \left[ \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} \left[ \int \frac{dp_{\perp}^2}{p_{\perp}^2} (A(\alpha_s) + B(\alpha_s)) \right] \right]^{\text{for DIS}}
\]

\[
A(\alpha_s) = A^{(1)} \frac{\alpha_s}{\pi} + A^{(2)} \left( \frac{\alpha_s}{\pi} \right)^2 + \ldots
\]

\[
B^{(1)} = -3C_F/2
\]

\[
A^{(2)} = \frac{1}{2} C_F \left( C_A \left( \frac{67}{18} - \frac{1}{6}\pi^2 \right) - \frac{5}{9} N_F \right) = \frac{1}{2} C_F K_{\text{CMW}}
\]

Replace

(for $z \to 1$: soft gluon limit):

\[
P_i(\alpha_s, z) = \frac{C_i \frac{\alpha_s}{\pi} \left( 1 + K_{\text{CMW}} \frac{\alpha_s}{2\pi} \right)}{1 - z}
\]
From \( \overline{\text{MS}} \) to \( \text{MC} \)


\[
P(\alpha_s, z) = \frac{\alpha_s}{2\pi} C_F \frac{A^{(1)}}{1 - z} + \left( \frac{\alpha_s}{\pi} \right)^2 \frac{A^{(2)}}{1 - z}
\]

Replace
(for \( z \to 1 \): soft gluon limit):

\[
P_i(\alpha_s, z) = \frac{C_i}{\pi} \left( \frac{\alpha_s}{2\pi} \right) \left( 1 + K_{\text{CMW}} \frac{\alpha_s}{2\pi} \right)
\]

\[
\alpha_s^{(\text{MC})} = \alpha_s^{(\overline{\text{MS}})} \left( 1 + K_{\text{CMW}} \frac{\alpha_s^{(\overline{\text{MS}})}}{2\pi} \right)
\]

\[
\Lambda_{\text{MC}} = \Lambda_{\text{MS}} \exp \left( \frac{K_{\text{CMW}}}{4\pi \beta_0} \right) \sim 1.57 \Lambda_{\text{MS}}
\]

Note also: used \( m_u^2 = p_T^2 = (1-z)Q^2 \)
Amati, Bassetto, Ciafaloni, Marchesini, Veneziano, 1980

Main Point:
Doing an uncompensated scale variation actually ruins this result
Nice to have all-orders solution

But it is only exact in the singular (soft & collinear) limits

→ gets the bulk of bremsstrahlung corrections right, but fails equally spectacularly: for hard wide-angle radiation: **visible, extra jets**

... which is exactly where fixed-order calculations work!

So combine them!

→ Matching Lectures by Stefan Höche
Z → 3 Jets

Size of NLO “K” factor over phase space

\[ y_{ij} = \frac{s_{ij}}{M_Z^2} \]

\[ y_{jk} = \frac{s_{jk}}{M_Z^2} \]

\[
\begin{align*}
(a) & \quad \mu_{PS} = \sqrt{s} \\
(b) & \quad \mu_{PS} = p_{\perp}
\end{align*}
\]
Z → 3 Jets  Size of NLO “K” factor over phase space

The “CMW” factor

\[ k_{CMW} = \exp \left( \frac{67 - 3\pi^2 - 10n_F/3}{2(33 - 2n_F)} \right) = \begin{cases} 
1.513 & n_F = 6 \\
1.569 & n_F = 5 \\
1.618 & n_F = 4 \\
1.661 & n_F = 3 
\end{cases} \]

Catani, Marchesini, Webber, NPB349

(b) \( \mu_{PS} = p_\perp \)

\[ \mu_{PS} = p_\perp, \text{ with CMW} \]
2 Loop: \( \alpha_s(M_Z)=0.12 \quad \Lambda_3 = 0.37 \quad \Lambda_4 = 0.32 \quad \Lambda_5 = 0.23 \)

1 Loop: \( \alpha_s(M_Z)=0.14 \quad \Lambda_3 = 0.37 \quad \Lambda_4 = 0.33 \quad \Lambda_5 = 0.26 \)

(In all cases, 5-flavor running is still used above \( m_t \))
DGLAP for Parton Density

\[
\frac{df_b(x, t)}{dt} = \sum_{a,c} \int \frac{dx'}{x'} f_a(x', t) \frac{\alpha_{abc}}{2\pi} P_{a\rightarrow bc}\left(\frac{x}{x'}\right)
\]

→ Sudakov for ISR

\[
\Delta(x, t_{\text{max}}, t) = \exp \left\{ - \int_{t}^{t_{\text{max}}} dt' \sum_{a,c} \int \frac{dx'}{x'} f_a(x', t') \frac{\alpha_{abc}(t')}{2\pi} P_{a\rightarrow bc}\left(\frac{x}{x'}\right) \right\}
\]

\[
= \exp \left\{ - \int_{t}^{t_{\text{max}}} dt' \sum_{a,c} \int dz \frac{\alpha_{abc}(t')}{2\pi} P_{a\rightarrow bc}(z) \frac{x' f_a(x', t')}{xf_b(x, t')} \right\},
\]
THE SHOWER OPERATOR

\[
\begin{align*}
\text{Born} \quad \frac{d\sigma_H}{d\mathcal{O}} \bigg|_{\text{Born}} &= \int d\Phi_H \ |M_H^{(0)}|^2 \, \delta(\mathcal{O} - \mathcal{O}(\{p\}_H)) \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad}
To ALL Orders

\[ S(\{p\}_x, \mathcal{O}) = \Delta(t_{\text{start}}, t_{\text{had}}) \delta(\mathcal{O} - \mathcal{O}(\{p\}_x)) \]

"Nothing Happens" → "Evaluate Observable"

\[ - \int_{t_{\text{start}}}^{t_{\text{had}}} dt \frac{d\Delta(t_{\text{start}}, t)}{dt} S(\{p\}_{X+1}, \mathcal{O}) \]

"Something Happens" → "Continue Shower"

All-orders Probability that nothing happens

\[ \Delta(t_1, t_2) = \exp \left( - \int_{t_1}^{t_2} dt \frac{d\mathcal{P}}{dt} \right) \]

(Exponentiation)

Analogous to nuclear decay

\[ N(t) \approx N(0) \exp(-ct) \]