Australia’s 4 deadliest animals:
- Horses (7/yr)
- Cows (3/yr)
- Dogs (3/yr)
- Roos (2/yr)

Monash University:
- 70,000 students (Australia’s largest uni)
- ~ 20km SE of Melbourne City Centre

School of Physics & Astronomy;
- 4 HEP theorists + post docs & students
This course covers:

Lecture 1: Foundations of MC Generators
Lecture 2: Parton Showers
Lecture 3: Jets and Confinement
Lecture 4: Physics at Hadron Colliders


It does not cover:

Simulation of BSM physics → Lectures by V Hirschi
Matching and Merging → Lectures by S Höche
Heavy Ions and Cosmic Rays → Lectures by K Werner
Event Generator Tuning → Lecture by H Schulz
+ many other (more specialised) topics such as: heavy quarks, hadron and τ decays, exotic hadrons, lattice QCD, spin/polarisation, low-x, elastic, …
1. Foundations of MC Generators
2. Parton Showers
3. Jets and Confinement
4. Physics at Hadron Colliders
MAKING PREDICTIONS

Scattering Experiments:

→ Integrate differential cross sections over specific phase-space regions

Predicted number of counts
= integral over solid angle

In particle physics:
Integrate over all quantum histories (+ interferences)

\[ N_{\text{count}}(\Delta \Omega) \propto \int_{\Delta \Omega} \frac{d\sigma}{d\Omega} \]
If event generators could talk:

Someone hold my drink while I approximate the amplitude (squared) for this …

(to all orders, + non-perturbative effects)

\[ \frac{d\sigma}{d\Omega} \]; how hard can it be?

… integrate it over a ~300-dimensional phase space

… and estimate the detector response
QCD in Event Generators
\[ \mathcal{L} = \bar{\psi}_q^i (i \gamma^\mu) (D_\mu)_{ij} \psi_j^j - m_q \bar{\psi}_q^i \psi_q^i - \frac{1}{4} F_{a \mu \nu} F^{a \mu \nu} \]

**Quark fields**

\[
\psi_j^q = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}
\]

**Covariant Derivative**

\[
D_{ij}^\mu = \delta_{ij} \partial_\mu - i g_s t_{ij}^a A^\mu_a
\]

\[ \Rightarrow \text{Feynman rules} \]

**Gell-Mann Matrices** \((t^a = \frac{1}{2} \lambda^a)\)

(Traceless and Hermitian)

\[
\begin{align*}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
\lambda^5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, & \lambda^6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda^8 &= \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{pmatrix}
\end{align*}
\]
A quark-gluon interaction

(= one term in sum over colours)

\[ \propto -\frac{i}{2} g_s \quad \bar{\psi}_q \gamma^\mu D_{ij}^\mu \psi_q \]

\[ = -\frac{i}{2} g_s \left( \begin{array}{ccc} 1 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \]

Gluon (adjoint) colour index \( \in [1,8] \)
Gluon Lorentz-vector index \( \in [0,3] \)

Fermion colour indices \( \in [1,3] \)
Fermion spinor indices \( \in [1,4] \)

Amplitudes Squared summed over colours \( \rightarrow \) traces over \( t \) matrices

\( \rightarrow \) **Colour Factors** (see literature, or back of these slides)
A gluon-gluon interaction (no equivalent in QED)

$$A^4_\nu(k_2)$$

$$A^6_\rho(k_1)$$

$$A^2_\mu(k_3)$$

(there is also a 4-gluon vertex, proportional to $g_s^2$)

Amplitudes Squared summed over colours $\rightarrow$ traces over $t$ matrices $\rightarrow$ **Colour Factors** (see literature, or back of these slides)
MC generators use a simple set of rules for “colour flow”

Based on “Leading Colour” \( 8 = 3 \otimes \overline{3} \oplus 1 \) (\( \Rightarrow \) valid to \( \sim 1/N_C^2 \sim 10\% \))

LC: represent gluons as outer products of triplet and antitriplet

"Strong Ordering", \( \alpha_s(p_\perp) \), "Coherence", "Recoils" [(E,p) cons.]

+ Mass effects: \( t, b, (c?) \) quarks, coloured resonances;
Spin effects (J cons; polarisation; spin correlations);
Corrections beyond LC (or LL)
Showers (can) generate lots of partons, $\mathcal{O}(10-100)$.

Colour Flow used to determine between which partons confining potentials arise

Example: $Z^0 \rightarrow qq$

Coherence of pQCD cascades $\rightarrow$ suppression of “overlapping” systems
$\rightarrow$ Leading-colour approximation pretty good

(LEP measurements in $e^+e^-\rightarrow W^+W^-\rightarrow$ hadrons confirm this (at least to order 10% ~ $1/N_c^2$))

Note: (much) more color getting kicked around in hadron collisions.
Signs that LC approximation is breaking down? $\rightarrow$ Lecture 4
Bjorken scaling:

To first approximation, QCD is **SCALE INvariant** (a.k.a. conformal)

Jets inside jets inside jets …

Fluctuations (loops) inside fluctuations inside fluctuations …

If the strong coupling didn’t “run”, this would be absolutely true (e.g., N=4 Supersymmetric Yang-Mills)

Since $\alpha_s$ only runs slowly (logarithmically) $\rightarrow$ can still gain insight from fractal analogy ($\rightarrow$ lecture 2 on showers)

Note: I use the terms “conformal” and “scale invariant” interchangeably

Strictly speaking, conformal (angle-preserving) symmetry is more restrictive than just scale invariance
The strong coupling is (one of) the main perturbative parameter(s) in event generators. It controls:

- The overall amount of QCD initial- and final-state radiation
- Strong-interaction cross sections (and resonance decays)
- The rate of (mini)jets in the underlying event

Example (for Final-State Radiation):

SHERPA: uses PDF or PDG value, with “CMW” translation
\( \alpha_s(m_Z) \) default = 0.118 (pp) or 0.1188 (LEP)
running order: default = 3-loop (pp) or 2-loop (LEP)
CMW scheme translation: default use ~ \( \alpha_s(p_T/1.6) \)
→ roughly 10% increase in the effective value of \( \alpha_s \)

will undershoot LEP 3-jet rate by ~ 10% (unless combined with NLO 3-jet ME)

PYTHIA: tuning to LEP 3-jet rate; requires ~ 20% increase
TimeShower:alphaSvalue default = 0.1365
TimeShower:alphaSorder default = 1
TimeShower:alphaSuseCMW default = off

Agrees with LEP 3-jet rate “out of the box”; but no guarantee tuning is universal.
Scale variation ~ uncertainty; why?

Scale dependence of calculated orders must be canceled by contribution from uncalculated ones (+ non-pert)

\[
\alpha_s(Q^2) = \alpha_s(m_Z^2) \left( \frac{1}{1 + b_0 \alpha_s(m_Z) \ln \frac{Q^2}{m_Z^2} + \mathcal{O}(\alpha_s^2)} \right)
\]

\[b_0 = \frac{11N_C - 2n_f}{12\pi}\]

\[
\rightarrow \alpha_s(Q_1^2) - \alpha_s(Q_2^2) = \alpha_s^2 b_0 \ln(Q_2^2/Q_1^2) + \mathcal{O}(\alpha_s^3)
\]

\[\rightarrow \text{Generates terms of higher order, proportional to what you already have } (|M|^2) \rightarrow \text{a first naive}^* \text{ way to estimate uncertainty}\]

*warning: some believe it is the only way … but be agnostic! Really a lower limit. There are other things than scale dependence …
Example: \( pp \to W + 3 \text{ jets} \)

1: \( MW \)
2: \( MW + \text{Sum}(|p_T|) \)
3: -“- (quadratically)
4: Geometric mean \( p_T \) (~shower)
5: Arithmetic mean \( p_T \)

If you have multiple QCD scales
→ variation of \( \mu_R \) by factor 2 in each direction not exhaustive!

Also consider functional dependence on each scale (+ \( N^{(n)}\text{LO} \) → some compensation)
QCD is more than just a perturbative expansion in $\alpha_s$ (and Perturbation theory is more than Feynman diagrams).

**Jets** $\leftrightarrow$ amplitude structures $\leftrightarrow$ fundamental quantum field theory / gauge theory. Precision jet (structure) studies. ➜ Lecture 2

**Strings** (strong gluon fields) $\leftrightarrow$ quantum-classical correspondence. String physics. Dynamics of confinement / hadronisation phase transition. ➜ Lecture 3

**Hadrons** $\leftrightarrow$ Spectroscopy (incl excited and exotic states), lattice QCD, (rare) decays, mixing, light nuclei. Hadron beams $\rightarrow$ MPI, diffraction, ... ➜ Lecture 4
Factorisation $\Rightarrow$ Fixed-order cross sections still useful.

In DIS, there is a formal proof (Collins, Soper, 1987)

Deep Inelastic Scattering (DIS)

(By “deep”, we mean $Q^2 \gg M_h^2$)

→ We really can write the cross section in factorised form:

$$
\sigma^{\ell h} = \sum_i \sum_f \int dx_i \int d\Phi_f \, f_{i/h}(x_i, Q_f^2) \frac{d\hat{\sigma}^{\ell i \rightarrow f}(x_i, \Phi_f, Q_F^2)}{dx_i \, d\Phi_f}
$$

- Sum over initial (i) and final (f) parton flavors
- $\Phi_f$ = Final-state phase space
- $f_{i/h}$ = PDFs
- Assumption: $Q^2 = Q_F^2$
- Differential partonic hard-scattering matrix element(s)
Why do we need PDFs, parton showers / jets, etc.? Why are Fixed-Order QCD matrix elements not enough?

F.O. QCD requires **Large scales** \( \Rightarrow \alpha_s \) small enough to be perturbative

\( \cdots \) cannot be used to address intrinsically soft physics such as minimum-bias or diffraction, but still OK for high-scale/hard processes

F.O. QCD requires **No scale hierarchies** \( \Rightarrow \alpha_s \ln(Q_i/Q_j) \) small

In the presence of scale hierarchies, propagator singularities integrate to logarithms (tomorrow’s lecture) which ruin fixed-order expansion.

**But!!!** we collide - and observe - hadrons, with *non-perturbative structure*, that participate in hard processes, whose scales are *hierarchically greater* than \( m_{\text{had}} \sim 1 \text{ GeV} \).

\[ \rightarrow \text{A Priori, no perturbatively calculable observables in QCD} \]
Why is Fixed Order QCD not enough? It requires all resolved scales \( \Lambda_{\text{QCD}} \) AND no large hierarchies.

PDFs: connect incoming hadrons with the high-scale process. 

Fragmentation Functions: connect high-scale process with final-state hadrons (each is a non-perturbative function modulated by initial- and final-state radiation).

\[
\frac{d\sigma}{dX} = \sum_{a,b} \sum_f \int \hat{X}_f \, f_a(x_a, Q_i^2) f_b(x_b, Q_i^2) \frac{d\hat{\sigma}_{ab\rightarrow f}(x_a, x_b, f, Q_i^2, Q_f^2)}{d\hat{X}_f} \, D(\hat{X}_f \rightarrow X, Q_i^2, Q_f^2)
\]

PDFs: needed to compute inclusive cross sections. 

FFs: needed to compute (semi-)exclusive cross sections.

In MCs: made exclusive as \textbf{initial-state radiation} + non-perturbative hadron (beam-remnant) structure (+ multiple parton-parton interactions).

In MCs: \textbf{resonance decays, final-state radiation}, hadronisation, hadron decays (+ final-state interactions?)

\textbf{Resummed pQCD}: All resolved scales \( \gg \Lambda_{\text{QCD}} \) AND \textbf{X Infrared Safe}

\( ^{\text{pQCD = perturbative QCD}} \)

Will take a closer look at both PDFs and final-state aspects (jets and showers) in the next lectures.
**Divide and Conquer** → Split the problem into many (nested) pieces

+ Quantum mechanics → Probabilities → Random Numbers

\[ P_{\text{event}} = P_{\text{hard}} \otimes P_{\text{dec}} \otimes P_{\text{ISR}} \otimes P_{\text{FSR}} \otimes P_{\text{MPI}} \otimes P_{\text{Had}} \otimes \ldots \]

**Hard Process & Decays:**
Use process-specific (N)LO matrix elements (e.g., \( gg \rightarrow H^0 \rightarrow \gamma \gamma \))
→ Sets “hard” resolution scale for process: \( Q_{\text{MAX}} \)

**ISR & FSR (Initial- & Final-State Radiation):**
Driven by differential (e.g., DGLAP) evolution equations, \( dP/dQ^2 \), as function of resolution scale; from \( Q_{\text{MAX}} \) to \( Q_{\text{HAD}} \sim 1 \text{ GeV} \)

**MPI (Multi-Parton Interactions)**
Protons contain lots of partons → can have additional (soft) parton-parton interactions → Additional (soft) “Underlying-Event” activity

**Hadronisation**
Non-perturbative modeling of partons → hadrons transition
THE MAIN WORKHORSES

PYTHIA (begun 1978)
Originated in hadronisation studies: Lund String model
Still significant emphasis on soft/non-perturbative physics

HERWIG (begun 1984)
Originated in coherence studies: angular-ordered showers
Cluster hadronisation as simple complement

SHERPA (begun ~2000)
Originated in ME/PS matching (CKKW-L)
Own variant of cluster hadronisation

+ Many more specialised:
Matrix-Element Generators, Matching/Merging Packages, Resummation packages,
Alternative QCD showers, Soft-QCD MCs, Cosmic-Ray MCs, Heavy-Ion MCs, Neutrino MCs, Hadronic interaction MCs (GEANT/FLUKA; for energies below $E_{\text{CM}} \sim 10$ GeV), (BSM) Model Generators, Decay Packages, …
MC: any technique that makes use of random sampling (to provide numerical estimates)
Prescribed for cases of complicated integrands/boundaries in high dimensions
MC: any technique that makes use of random sampling (to provide numerical estimates)

Prescribed for cases of complicated integrands/boundaries in high dimensions

**Example: Integrate f(x)**

1. Compute area of box (you can do it!)
2. Throw random (x,y) points uniformly inside box
3. If \( y < f(x) \): accept (blue); else reject (red)
4. After \( N_{\text{tot}} \) throws, you have an estimate
   \[
   \int_{x_{\text{min}}}^{x_{\text{max}}} f(x) \, dx \sim A_{\text{box}} \frac{N_{\text{blue}}}{N_{\text{tot}}}
   \]
5. Central limit theorem \( \Rightarrow \) converges to \( A_{\text{blue}} \)

**Recap Convergence:**

- **Calculus:** \{A\} converges to B if \( n \) exists for which \( |A_i - B| < \varepsilon \), for any \( \varepsilon > 0 \)
- **Monte Carlo:** \{A\} converges to B if \( n \) exists for which the probability for \( |A_i - B| < \varepsilon \), is \( P \), for any \( P \in (0,1) \) for any \( \varepsilon > 0 \)

“This risk, that convergence is only given with a certain probability, is inherent in Monte Carlo calculations and is the reason why this technique was named after the world’s most famous gambling casino.” —F. James, MC theory and practice
Example: Integrate $f(x)$

Could also have used standard 1D num. int. (e.g., “Fixed-Grid”: Trapezoidal rule, Simpson’s rule …) → typically faster convergence in 1D but few general optimised methods in 2D; none beyond 3D & convergence rate becomes worse …

The convergence rate of MC remains the stochastic $1/\sqrt{n}$ independent of dimension* !

*) You still need to worry about variance; physics has lots of peaked/singular functions → adaptive sampling (or stratification)

### Numerical Integration: Relative Uncertainty

<table>
<thead>
<tr>
<th></th>
<th>n_{eval} / bin</th>
<th>One Dimension Conv. Rate</th>
<th>D Dimensions Conv. Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trapezoidal Rule (2-point)</td>
<td>$2^D$</td>
<td>$1/n^2$</td>
<td>$1/n^{2/D}$</td>
</tr>
<tr>
<td>Simpson’s Rule (3-point)</td>
<td>$3^D$</td>
<td>$1/n^4$</td>
<td>$1/n^{4/D}$</td>
</tr>
<tr>
<td>Monte Carlo</td>
<td>1</td>
<td>$1/n^{1/2}$</td>
<td>$1/n^{1/2}$</td>
</tr>
</tbody>
</table>

+ optimisations (stratification, adaptation), **iterative solutions** (Markov-Chain Monte Carlo)
JUSTIFICATION:
MC CAN PROVIDE PERFECT ACCURACY, WITH STOCHASTIC PRECISION

1. Law of large numbers (MC is accurate)
   For a function, \( f \), of random variables, \( x_i \),
   \[
   \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx
   \]
   (note: in real world, we only deal with approximations to Nature’s \( f(x) \rightarrow \) less than perfect accuracy)

2. Central limit theorem (MC precision is stochastic: \( 1/\sqrt{n} \))
   The sum of \( n \) independent random variables (of finite expectations and variances) is asymptotically Gaussian
   (no matter how the individual random variables are distributed)
   For finite \( n \):
   The Monte Carlo estimate is Gauss distributed around the true value \( \rightarrow \) with \( 1/\sqrt{n} \) precision
   In other words: MC stat unc same as for data

For infinite \( n \):
Monte Carlo is a consistent estimator

Variance \( V = (x_2 - x_1) \int_{x_1}^{x_2} [f(x)]^2 \, dx - \left( \int_{x_1}^{x_2} f(x) \, dx \right)^2 \)
PEAKED FUNCTIONS

Precision on integral dominated by the points with \( f \approx f_{\text{max}} \) (i.e., peak regions)

\[ \rightarrow \text{slow convergence if high, narrow peaks} \]
STRATIFIED SAMPLING

→ Make it twice as likely to throw points in the peak

Choose:

For:

- $[0,1] \rightarrow$ Region A
- $[1,2] \rightarrow$ Region B
- $6*R_1 \in [2,4] \rightarrow$ Region C
- $[4,5] \rightarrow$ Region D
- $[5,6] \rightarrow$ Region E

→ faster convergence for same number of function evaluations


Can even design algorithms that do this automatically as they run (not covered here)

→ Adaptive sampling
Note: if several peaks: do **multi-channel importance sampling** (~ competing random processes)

→ or throw points according to some smooth peaked function for which you have, or can construct, a random number generator (here: Gauss)

Any MC generator contains LOTS of examples of this.

(+ some generic algorithms though generally never as good as dedicated ones: e.g., VEGAS algorithm)
WHY DOES THIS WORK?

1) You are inputting knowledge: obviously need to know where the peaks are to begin with … (say you know, e.g., the location and width of a resonance or singularity)

2) Stratified sampling increases efficiency by combining n-point quadrature with the MC method, with further gains from adaptation

3) Importance sampling:

\[
\int_{a}^{b} f(x)\,dx = \int_{a}^{b} \frac{f(x)}{g(x)}\,dG(x)
\]

Effectively does flat MC with changed integration variables

Fast convergence if \(f(x)/g(x) \approx 1\)

Flat sampling in \(x\) \(\rightarrow\) Flat sampling in \(G(x)\)
Complicated Function:

**Time-dependent**

Traffic density during day, week-days vs week-ends  
(i.e., NON-TRIVIAL TIME EVOLUTION OF SYSTEM)

**No two pedestrians are the same**

Need to compute probability for each and sum  
(SIMULATES HAVING SEVERAL DISTINCT TYPES OF “Evolvers”)

**Multiple outcomes (ignored for today):**

Hit → keep walking, or go to hospital?  
Multiple hits = Product of single hits, or more complicated?
Approximate Traffic

Simple overestimate:
- highest recorded density of most careless drivers,
- driving at highest recorded speed

... 

Approximate Pedestrian

by most completely reckless and accident-prone person (e.g., MCnet student wandering the streets lost in thought after these lectures …)

This extreme guess will be the equivalent of a simple area (~integral) we can calculate:
Off we go…

Throw random accidents according to:

\[ R = \int_{t_0}^{t} \int_{\text{Area}} d^2x \sum_{i=1}^{n_{\text{ped}}} \alpha_i(x, t') \rho_i(x, t') \rho_c(x, t') \]

- Uniformly distributed random number \( \in [0,1] \)
- \( t_0 : \text{starting time} \)
- \( t : \text{time of accident} \)
- Sum over Pedestrians

\[ R_{\text{trial}} = \left( t_{\text{trial}} - t_0 \right) \left( \pi r_{\text{max}}^2 \right) \alpha_{\text{max}} n_{\text{ped}} \rho_{c\text{max}} \]

- Solver for \( t_{\text{trial}}(R_{\text{trial}}) \)
- Larger trial area with simple boundary (in this case, circle)
- Hit rate for most accident-prone pedestrian with worst driver
- Rush-hour density of cars

(Also generate trial \( x \), e.g., uniformly in circular area around Lund)
(Also generate trial \( i \); a random pedestrian gets hit)

(note: this generator is unordered; not asking whether that pedestrian was already hit earlier…)
Now you have a trial. Veto the trial if generated $x$ is outside desired physical boundary. If inside, accept trial hit $(i,x,t)$ with probability (exactly equivalent to when we coloured points blue [accept] or red [reject])

\[
\text{Prob(accept) = } \frac{\alpha_i(x, t) \rho_i(x, t) \rho_c(x, t)}{\alpha_{\text{max}} \rho_{\text{cmax}}}
\]

Using the following:

- $\rho_c$: actual density of cars at location $x$ at time $t$
- $\rho_i$: actual density of student $i$ at location $x$ at time $t$
- $\alpha_i$: The actual “hit rate” (OK, not really known, but could fit to past data: “tuning”)

→ True number = number of accepted hits (caveat: we didn’t really treat multiple hits … → Sudakovs & Markov Chains; tomorrow)
Take your system

Generate a “trial” (event/decay/interaction/… )

Not easy to generate random numbers distributed according to exactly the right distribution?

May have complicated dynamics, interactions …

→ use a simpler “trial” overestimating distribution

Flat with some stratification

Or importance sample with simple overestimating function (for which you can ~ easily generate random numbers)
SUMMARY: HOW WE DO MONTE CARLO

Take your system

Generate a “trial” (event/decay/interaction/…)

Accept trial with probability \( f(x)/g(x) \)
- \( f(x) \) contains all the complicated dynamics
- \( g(x) \) is the simple trial function

If accept: replace with new system state
If reject: keep previous system state

And keep going: generate next trial …

no dependence on \( g(x) \) in final result - only affects convergence rate
SUMMARY: HOW WE DO MONTE CARLO

Take your system

Generate a “trial” (event/decay/interaction/…)

Accept trial with probability $f(x)/g(x)$
  
  $f(x)$ contains all the complicated dynamics
  
  $g(x)$ is the simple trial function

If accept: replace with new system state

If reject: keep previous system state

no dependence on $g(x)$ in final result - only affects convergence rate

Sounds deceptively simple, but …

**with it, you can integrate**

arbitrarily complicated functions (*and chains of nested functions*),

over arbitrarily complicated regions, in arbitrarily many dimensions …

And keep going: generate next trial …
A Psychological Tip

Whenever you're called on to make up your mind, and you're hampered by not having any, the best way to solve the dilemma, you'll find, is simply by spinning a penny.

No -- not so that chance shall decide the affair while you're passively standing there moping; but the moment the penny is up in the air, you suddenly know what you're hoping.

[Piet Hein, Danish scientist, poet & friend of Niels Bohr]
Extra Slides
I will not tell you how to write a Random-number generator. (For that, see the references in the writeup.)

Instead, I **assume** that you can write a computer code and link to a random-number generator, from a library

E.g., ROOT includes one that you can use if you like. PYTHIA also includes one

From the PYTHIA 8 HTML documentation, under "Random Numbers":

```
Random numbers R uniformly distributed in 0 < R < 1 are obtained with
Pythia8::Rndm::flat();
```

+ Other methods for exp, x*exp, 1D Gauss, 2D Gauss.
Example 1: simple function (=constant); complicated boundary

Assume you know the area of this shape:

$$\pi R^2$$ (an overestimate)

Now get a few friends, some balls, and throw random shots inside the circle (PS: be careful to make your shots truly random)

Count how many shots hit the shape inside and how many miss

$$A \approx \frac{N_{hit}}{N_{miss}} \times \pi R^2$$

Earliest Example of MC calculation: Buffon’s Needle (1777) to calculate $$\pi$$

G. Leclerc, Comte de Buffon (1707-1788)
Colour Factors

Processes involving coloured particles have a “colour factor”.
It counts the enhancement from the sum over colours.

(average over incoming colours → can also give suppression)
Colour Factors

Processes involving coloured particles have a “colour factor”. It counts the enhancement from the sum over colours.

\[
\sum_{\text{colours}} |M|^2 = \delta_{ij} \quad \propto \delta_{ij} \delta_{ji}^* \quad = \text{Tr}[\delta_{ij}] \quad = NC
\]

\(i,j \in \{R,G,B\}\)
Colour Factors

Processes involving coloured particles have a “colour factor”. It counts the enhancement from the sum over colours.

(average over incoming colours → can also give suppression)
CROSSINGS

\[ e^+ e^- \rightarrow \gamma^*/Z \rightarrow q\bar{q} \]
(Hadronic Z Decay)

\[ q\bar{q} \rightarrow \gamma^*/Z \rightarrow \ell^+\ell^- \]
(Drell & Yan, 1970)

\[ \ell_q \gamma^*/Z \rightarrow \ell_q \]
(DIS)

\[
\begin{align*}
\text{Color Factor:} & \\
\text{Tr}[\delta_{ij}] &= N_C \\
\frac{1}{N_C^2} \text{Tr}[\delta_{ij}] &= \frac{1}{N_C} \\
\frac{1}{N_C} \text{Tr}[\delta_{ij}] &= 1
\end{align*}
\]
Colour Factors

Processes involving coloured particles have a “colour factor”. It counts the enhancement from the sum over colours.

(average over incoming colours → can also give suppression)

\[
\sum \text{colours} \quad |M|^2 = \delta_{ij} \quad \text{\textit{Z}→3\ jets} \quad \alpha \quad \delta_{ij} t^a_{jk} t^a_{k\ell} \delta_{\ell i} = \text{Tr}\{t^a t^a\} = \frac{1}{2} \text{Tr}\{\delta\} = 4
\]
Colour factors squared produce traces

\[ \text{Tr}(t^A t^B) = T_R \delta^{AB}, \quad T_R = \frac{1}{2} \]

\[ \sum_A t^A_{ab} t^A_{bc} = C_F \delta_{ac}, \quad C_F = \frac{N_c^2 - 1}{2 N_c} = \frac{4}{3} \]

\[ \sum_{C,D} f^{ACD} f^{BCD} = C_A \delta^{AB}, \quad C_A = N_c = 3 \]

\[ t^A_{ab} t^A_{cd} = \frac{1}{2} \delta_{bc} \delta_{ad} - \frac{1}{2N_c} \delta_{ab} \delta_{cd} \quad \text{(Fierz)} \]

\[ \text{Example Diagram} \]

(from ESHEP lectures by G. Salam)
Real QCD isn’t conformal

The coupling runs logarithmically with the energy scale

\[ Q^2 \frac{\partial \alpha_s}{\partial Q^2} = \beta(\alpha_s) \]

\[ \beta(\alpha_s) = -\alpha_s^2 (b_0 + b_1 \alpha_s + b_2 \alpha_s^2 + \ldots) , \]

\[ b_0 = \frac{11C_A - 2n_f}{12\pi} \quad b_1 = \frac{17C_A^2 - 5C_A n_f - 3C_F n_f}{24\pi^2} = \frac{153 - 19n_f}{24\pi^2} \]

1-Loop \( \beta \) function coefficient

2-Loop \( \beta \) function coefficient

Asymptotic freedom in the ultraviolet

Confinement (IR slavery?) in the infrared
Multi-Scale Exercise

If needed, can convert from multi-scale to single-scale

\[
\alpha_s(\mu_1)\alpha_s(\mu_2) \cdots \alpha_s(\mu_n) = \prod_{i=1}^{n} \alpha_s(\mu) \left( 1 + b_0 \alpha_s \ln \left( \frac{\mu^2}{\mu_i^2} \right) + \mathcal{O}(\alpha_s^2) \right)
\]

\[
= \alpha_s^n(\mu) \left( 1 + b_0 \alpha_s \ln \left( \frac{\mu^{2n}}{\mu_1^2 \mu_2^2 \cdots \mu_n^2} \right) + \mathcal{O}(\alpha_s^2) \right)
\]

by taking geometric mean of scales
Phase Space Generation

\[\sigma = \frac{1}{2s} \int |\mathcal{M}|^2 \, d\Pi_n(\sqrt{s})\]

\[\Gamma = \frac{1}{2M} \int |\mathcal{M}|^2 \, d\Pi_n(M)\]

- Phase space:

\[d\Pi_n(M) = \left[ \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3(2E_i)} \right] (2\pi)^4 \delta^4 \left( p_0 - \sum_{i=1}^{n} p_i \right)\]

- Two-body easy:

\[d\Pi_2(M) = \frac{1}{8\pi} \frac{2p}{M} \frac{d\Omega}{4\pi}\]
• Other cases by recursive subdivision:

\[
d\prod_n(M) = \frac{1}{2\pi} \int_0^{(M-m)^2} d\prod_2(M) \prod_{n-1}(m_x)\]

• Or by ‘democratic’ algorithms: RAMBO, MAMBO
  Can be better, but matrix elements rarely flat.
Particle Decays

- Simplest example

  e.g. top quark decay:

\[
|M|^2 = \frac{1}{2} \left( \frac{8\pi\alpha}{\sin^2 \theta_W} \right)^2 \frac{p_t \cdot p_\ell \ p_b \cdot p_\nu}{(m_W^2 - M_W^2)^2 + \Gamma_W^2 M_W^2}
\]

\[
\Gamma = \frac{1}{2M} \frac{1}{128\pi^3} \int |M|^2 \, dm_W^2 \left( 1 - \frac{m_W^2}{M^2} \right) \frac{d\Omega}{4\pi} \frac{d\Omega_W}{4\pi}
\]

Breit-Wigner peak of W very strong - must be removed by importance sampling:

\[
m_W^2 \rightarrow \arctan \left( \frac{m_W^2 - M_W^2}{\Gamma_W M_W} \right)
\]