Interleaved Evolution with NLO- and Helicity-Amplitudes

Peter Skands
(CERN TH)
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Now entering era of precision studies

+ huge amount of other physics studies:

# of journal papers so far: 225 ATLAS, 195 CMS, 83 LHCb, 62 ALICE

Some of these are already, or will ultimately be, theory limited
Why?

Precision = Clarity, in our vision of the Terascale

Searching towards lower cross sections, the game gets harder
+ Intense scrutiny (after discovery): precision = information

Theory task: invest in precision
(+ lots of interesting structures in QFT, can compare to data, ...)

This talk: a new formalism for highly accurate collider-physics calculations + some future perspectives

+ huge amount of other physics studies:

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How?

**Fixed Order Perturbation Theory:**
- Problem: limited orders

**Parton Showers:**
- Problem: limited precision

“Matching”: Best of both Worlds?
- Problem: stitched together, slow, limited orders

**Interleaved pQCD**
- Infinite orders, high precision, fast
The Problem of Bremsstrahlung

Jet Event at 2.36 TeV Collision Energy
2009-12-14, 04:30 CET, Run 142308, Event 482137
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Accelerated Charges

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Associated field (fluctuations) continues
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The Problem of Bremsstrahlung

The harder they get kicked, the harder the fluctuations that continue to become strahlung.
Most bremsstrahlung is driven by divergent propagators → simple structure

Amplitudes factorize in singular limits (→ universal “conformal” or “fractal” structure)

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Partons ab → “collinear”:

\[ P(z) = \text{Altarelli-Parisi splitting kernels, with } z = \text{energy fraction} = \frac{E_a}{E_a+E_b} \]

\[ |M_{F+1}(\ldots, a, b, \ldots)|^2 \xrightarrow{a \parallel b} g_s^2C \frac{P(z)}{2(p_a \cdot p_b)} |M_F(\ldots, a+b, \ldots)|^2 \]

Jets = Fractals

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Gluon j → "soft":

$$|\mathcal{M}_{F+1}(\ldots, i, j, k, \ldots)|^2 \xrightarrow{\text{soft}} g_s^2 C \frac{(p_i \cdot p_k)}{(p_i \cdot p_j)(p_j \cdot p_k)} |\mathcal{M}_F(\ldots, i, k, \ldots)|^2$$

+ scaling violation: $g_s^2 \rightarrow 4\pi\alpha_s(Q^2)$

**Jets = Fractals**

- **Most bremsstrahlung** is driven by divergent propagators $\rightarrow$ simple structure
- **Amplitudes factorize in singular limits** ($\rightarrow$ universal “conformal” or “fractal” structure)

**Partons** $ab \rightarrow$ **“collinear”:**

$$|\mathcal{M}_{F+1}(\ldots, a, b, \ldots)|^2 \propto \frac{a!b!}{2(p_a \cdot p_b)} \mathcal{P}(z) |\mathcal{M}_F(\ldots, a + b, \ldots)|^2$$

**Gluon** $j \rightarrow$ **“soft”:**

$$|\mathcal{M}_{F+1}(\ldots, i, j, k, \ldots)|^2 \propto \frac{(p_i \cdot p_k)}{(p_i \cdot p_j)(p_j \cdot p_k)} |\mathcal{M}_F(\ldots, i, k, \ldots)|^2$$

+ scaling violation: $g_s^2 \rightarrow 4\pi\alpha_s(Q^2)$

**Coherence** $\rightarrow$ Parton $j$ really emitted by $(i,k)$ “colour antenna”


Can apply this many times $\rightarrow$ nested factorizations
**Factorization** → Split the problem into many (nested) pieces

+ Quantum mechanics → Probabilities → Random Numbers

\[
P_{\text{event}} = P_{\text{hard}} \otimes P_{\text{dec}} \otimes P_{\text{ISR}} \otimes P_{\text{FSR}} \otimes P_{\text{MPI}} \otimes P_{\text{Had}} \otimes \ldots
\]

**Hard Process & Decays:**

Use (N)LO matrix elements

→ Sets “hard” resolution scale for process: \(Q_{\text{MAX}}\)

**ISR & FSR (Initial & Final-State Radiation):**

Altarelli-Parisi equations → differential evolution, \(dP/dQ^2\), as function of resolution scale; run from \(Q_{\text{MAX}}\) to ~ 1 GeV (More later)

**MPI (Multi-Parton Interactions)**

Additional (soft) parton-parton interactions: LO matrix elements

→ Additional (soft) “Underlying-Event” activity (Not the topic for today)

**Hadronization**

Non-perturbative model of color-singlet parton systems → hadrons
Unitarity (KLN):
Singular structure at loop level must be equal and opposite to tree level

$\rightarrow$ Virtual (loop) correction:

$$2\text{Re}[M_F^{(0)} M_F^{(1)*}] = -g_s^2 N_C |M_F^{(0)}|^2 \int \frac{ds_{ij} ds_{jk}}{16\pi^2 s_{ij} s_{jk}} \left( \frac{2s_{ik}}{s_{ij}s_{jk}} + \text{less singular terms} \right)$$

Loop = $- \text{Int(Tree)} + F$

Neglect $F \rightarrow$ Leading-Logarithmic (LL) Approximation
**Unitarity** (KLN):

Singular structure at loop level must be equal and opposite to tree level

\[ \rightarrow \text{Virtual (loop) correction:} \]

\[
2\text{Re}[\mathcal{M}_F^{(0)} \mathcal{M}_F^{(1)*}] = -g_s^2 N_C \left| \mathcal{M}_F^{(0)} \right|^2 \int \frac{d^4 s_{ij} d^4 s_{jk}}{16\pi^2 s_{ijk}} \left( \frac{2s_{ik}}{s_{ij}s_{jk}} + \text{less singular terms} \right)
\]

**Realized by Event evolution** in \( Q = \text{fractal scale} \) (virtuality, \( p_T \), formation time, ...)

- **Resolution scale**
  \( t = \ln(Q^2) \)

- **Probability to remain “unbranched” from \( t_0 \) to \( t \)**
  \[
  \frac{N_F(t)}{N_F(t_0)} = \Delta_F(t_0, t) = \exp \left( -\int \frac{d\sigma_{F+1}}{d\sigma_F} \right)
  \]

- **Approximation to Real Emissions**

- **Approximation to Loop Corrections**

**Loop** = - \( \text{Int(Tree)} + F \)

Neglect \( F \) → **Leading-Logarithmic (LL) Approximation**
Start from an arbitrary lowest-order process (green = QFT amplitude squared)

**Parton showers** generate the bremsstrahlung terms of the rest of the perturbative series (yellow = fractal with scaling violation)

**Universality (scaling)**

Jet-within-a-jet-within-a-jet-...

**Unitarity**

Cancellation of real & virtual singularities

**Exponentiation**

fluctuations within fluctuations
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Exponentiation
fluctuations within fluctuations

But ≠ full QCD! Only LL Approximation.
Jack of All Orders, Master of None?

“Good” Shower(s) \( \rightarrow \) Dominant all-orders structures

But what about all these unphysical choices?

Renormalization Scales (for each power of \( \alpha_s \))

The choice of shower evolution “time” \( \sim \) Factorization Scale(s)

The radiation/antenna/splitting functions (hard jets are non-singular)

Recoils (kinematics maps, \( d\Phi_{n+1}/d\Phi_n \))

The infrared cutoff contour (hadronization cutoff)

Nature does not depend on them \( \rightarrow \) vary to estimate uncertainties

Problem: existing approaches vary only one or two of these choices

1. Systematic Variations
   \( \rightarrow \) Comprehensive Theory
   Uncertainty Estimates

2. Higher-Order Corrections
   \( \rightarrow \) Systematic Reduction of Uncertainties
Including LO Matrix Elements

Conceptual Example of Current Approaches: MLM-like “Slicing”:
Use ME for $p_T > p_{T\text{match}}$; Use PS for $p_T < p_{T\text{match}}$

**Born**
- Compute inclusive $\sigma_B$
- Generate $d\sigma_B$ Phase Space
- Shower
- Reject if jet(s) > $p_{T\text{match}}$
  → retain Sudakov fraction
- → Exclusive $\sigma_B(p_{T\text{match}})$
- Unweight (incl PDFs, $\alpha_s$)

**Born + 1**
- Compute incl $\sigma_{B+1}(p_{T\text{match}})$
- Generate $d\sigma_{B+1}$ Phase Space
- Shower
- Reject if jet(s) > $p_{T\text{match}}$
  → retain Sudakov fraction
- → Exclusive $\sigma_{B+1}(p_{T\text{match}})$
- Unweight (incl PDFs, $\alpha_s$)

**Born + 2**
- Compute incl $\sigma_{B+2}(p_{T\text{match}})$
- Generate $d\sigma_{B+2}$ Phase Space
- Shower
- Reject if jet(s) > $p_{T2}$
  → retain Sudakov fraction
- → Inclusive $\sigma_{B+2}$
- Unweight (incl PDFs, $\alpha_s$)

Fixed Order is starting point. Treats each multiplicity as a separate calculation. Inefficiencies can enter in PS generation, Rejection, and Unweighting Steps.
Ask:

Is it possible to interpret the all-orders structure that a shower generates as a trial distribution for a more precise evolution?

Would essentially amount to using a QCD shower as your (only) phase space generator, on top of which fixed-order amplitudes are imprinted as (unitary and finite) multiplicative corrections.
Changing Paradigm

Ask:

Is it possible to interpret the all-orders structure that a shower generates as a trial distribution for a more precise evolution?

Would essentially amount to using a QCD shower as your (only) phase space generator, on top of which fixed-order amplitudes are imprinted as (unitary and finite) multiplicative corrections.

Answer:

Used to be no.

First order worked out in the eighties (Sjöstrand, also used in POWHEG), but higher-order expansions rapidly became too complicated.
Based on antenna factorization

- of Amplitudes (exact in both soft and collinear limits)
- of Phase Space (LIPS : 2 on-shell $\rightarrow$ 3 on-shell partons, with $(E,p)$ cons)
Based on antenna factorization
- of Amplitudes (exact in both soft and collinear limits)
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Resolution Time
Infinite family of continuously deformable $Q_E$
Special cases: transverse momentum, dipole mass, energy

Radiation functions
Arbitrary non-singular coefficients, $ant_i$
+ Massive antenna functions for massive fermions $(c, b, t)$

Kinematics maps
Formalism derived for arbitrary 2→3 recoil maps, $K_{3→2}$
Default: massive generalization of Kosower’s antenna maps

vincia.hepforge.org
Idea:
Start from quasi-conformal all-orders structure (approximate)
Impose exact higher orders as finite multiplicative corrections
Truncate at fixed scale (rather than fixed order)
Bonus: low-scale partonic events → can be hadronized

Problems:
Traditional parton showers are history-dependent (non-Markovian)
→ Number of generated terms grows like \(2^N N!\)
+ Dead zones and complicated expansions

Solution: (MC)² : Monte-Carlo Markov Chain
Markovian Antenna Showers (VINCI)
→ Number of generated terms grows like \(N\)
+ exact phase space & simple expansions

LO: Giele, Kosower, Skands, PRD 84 (2011) 054003
New: Markovian pQCD

Start at Born level

\[ |M_F|^2 \]

\[ \begin{array}{cccc}
+0 & +1 & +2 & +3 \\
\hline
+0 & \text{Empty} & \text{Empty} & \text{Empty} \\
+1 & \text{Empty} & \text{Empty} & \text{Empty} \\
+2 & \text{Empty} & \text{Empty} & \text{Empty} \\
+3 & \text{Empty} & \text{Empty} & \text{Empty} \\
\end{array} \]

The VINCIA Code + PYTHIA 8

“Higher-Order Corrections To Timelike Jets”
GeeKS: Giele, Kosower, Skands, PRD 84 (2011) 054003
HEL: Larkoski, Lopez-Villarejo, Skands, PRD 87 (2013) 054033

\[ pQCD : \text{perturbative QCD} \]
Start at Born level

$|M_F|^2$

Generate "shower" emission

$|M_{F+1}|^2 \sim \sum_{i \in \text{ant}} a_i |M_F|^2$

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PYTHIA 8
New: Markovian pQCD

Start at Born level
\[ |M_F|^2 \]

Generate "shower" emission
\[ |M_{F+1}|^2 \approx \sum_{i \in \text{ant}} a_i |M_F|^2 \]

Correct to Matrix Element
\[ a_i \rightarrow \frac{|M_{F+1}|^2}{\sum a_i |M_F|^2} a_i \]

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Unitarity of Shower
Virtual = - \int \text{Real}

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\[ |M_F|^2 \rightarrow |M_F|^2 + 2 \text{Re}[M_F^1 M_F^0] + \int \text{Real} \]

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Cutting Edge:
Embedding virtual amplitudes
= Next Perturbative Order
→ Precision Monte Carlos

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**Helicities**

Larkoski, Peskin, PRD 81 (2010) 054010
Larkoski, Lopez-Villarejo, Skands, PRD 87 (2013) 054033

**Traditional parton showers** use the standard Altarelli-Parisi kernels, \( P(z) = \text{helicity sums/averages over:} \)

\[
\begin{array}{c|cccc}
P(z) & ++ & -+ & +- & -- \\
g_+ \rightarrow gg & \frac{1}{z}(1-z) & \frac{(1-z)^3}{z} & \frac{z^3}{(1-z)} & 0 \\
g_+ \rightarrow q\bar{q} & - & \frac{(1-z)^2}{z} & z^2 & - \\
q_+ \rightarrow gg & \frac{1}{1-z} & - & \frac{z^2}{1-z} & - \\
q_+ \rightarrow g\bar{g} & \frac{1}{z} & \frac{(1-z)^2}{z} & - & - \\
\end{array}
\]

**Generalize** these objects to dipole-antennae

E.g.,

\( q\bar{q} \rightarrow qg\bar{q} \)

\[
\begin{align*}
++ & \rightarrow +++ & \text{MHV} \\
++ & \rightarrow +-- & \text{NMHV} \\
+- & \rightarrow +-- & \text{P-wave} \\
+- & \rightarrow +-- & \text{P-wave}
\end{align*}
\]

→ Can trace helicities through shower

→ Eliminates contribution from unphysical helicity configurations

→ Can match to individual helicity amplitudes rather than helicity sum

→ Fast! (gets rid of another factor \( 2^N \))
Flat phase-space scan. $H^0 \rightarrow q\bar{q} + ng$. Size of helicity contributions.
Helicity Contributions

Flat phase-space scan. $H^0 \rightarrow qq + ng$. Size of helicity contributions.

Distribution of PS/ME ratio (summed over helicities)

Vincia shower already quite close to ME → small corrections

Note: precision not greatly improved by helicity dependence
Helicity Contributions

Flat phase-space scan. $H^0 \to q\bar{q} + ng$. Size of helicity contributions.

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Larkoski, Lopez-Villarejo, Skands, PRD 87 (2013) 054033

### Speed

**1. Initialization time**
(to pre-compute cross sections and warm up phase-space grids)

10000

1000

100

10

1

0.1

2

3

4

5

6

SHERPA+COMIX

PYTHIA+VINCIA

**2. Time to generate 1000 events**
(Z → partons, fully showered & matched. No hadronization.)

1000

100

10

1

0.1

2

3

4

5

6

SHERPA (CKKW-L)

VINCIA (GKS)

unpolarized

polarized

Hadronization

Time (LEP)

Z → n : Number of Matched Legs

Z → uds cb ; Hadronization OFF ; ISR OFF ; udsc MASSLESS ; b MASSIVE ; ECM = 91.2 GeV ; Q_{match} = 5 GeV

SHERPA 1.4.0 (+COMIX) ; PYTHIA 8.1.65 ; VINCIA 1.0.29 + MADGRAPH 4.4.26 ;

gcc/gfortran v 4.7.1 -O2 ; single 3.06 GHz core (4GB RAM)
Pedagogical Example: $Z^0 \rightarrow q\bar{q}$ First Order ($\sim$POWHEG)

**Fixed Order:** Exclusive 2-jet rate (2 and only 2 jets), at $Q = Q_{\text{had}}$

\[
\begin{align*}
&= |M_0^0|^2 \left( 1 + \frac{2 \text{Re}[M_0^0 M_1^1^*]}{|M_0^0|^2} + \int_0^{Q_{\text{had}}^2} d\Phi_{\text{ant}} g_s^2 C A_{g/q\bar{q}} \right) \\
&= \frac{|M_1^0|^2}{|M_0^0|^2}
\end{align*}
\]
Pedagogical Example: $Z^0 \to q\bar{q}$ First Order (~POWHEG)

**Fixed Order:** Exclusive 2-jet rate (2 and only 2 jets), at $Q = Q_{\text{had}}$

$$|M_0^0|^2 \left(1 + \frac{2 \text{Re}[M_0^0 M_0^1^*]}{|M_0^0|^2} + \int_0^{Q_{\text{had}}^2} d\Phi_{\text{ant}} g_s^2 C A_{g/q\bar{q}} \right)$$

- Born
- Virtual
- Unresolved Real

$$= \frac{|M_1^0|^2}{|M_0^0|^2}$$

**LO Vincia:** Exclusive 2-jet rate (2 and only 2 jets), at $Q = Q_{\text{had}}$

$$|M_0^0|^2 \Delta(s, Q_{\text{had}}^2) = |M_0^0|^2 \left(1 - \int_0^s d\Phi_{\text{ant}} g_s^2 C A_{g/q\bar{q}} + O(\alpha_s^2) \right)$$

- Born
- Sudakov
- Approximate Virtual + Unresolved Real
Pedagogical Example: $Z^0 \rightarrow q\bar{q}$ First Order (~POWHEG)

**Fixed Order:** Exclusive 2-jet rate (2 and only 2 jets), at $Q = Q_{\text{had}}$

\[
|M_0^0|^2 \left( 1 + \frac{2 \text{Re}[M_0^0 M_0^{1*}]}{|M_0^0|^2} + \int_0^{Q_{\text{had}}^2} d\Phi_{\text{ant}} g_s^2 C A_{g/q\bar{q}} \right) = \frac{|M_1^0|^2}{|M_0^0|^2}
\]

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**LO Vincia:** Exclusive 2-jet rate (2 and only 2 jets), at $Q = Q_{\text{had}}$

\[
|M_0^0|^2 \Delta(s, Q_{\text{had}}^2) = |M_0^0|^2 \left( 1 - \int_{Q_{\text{had}}^2}^{s} d\Phi_{\text{ant}} g_s^2 C A_{g/q\bar{q}} + O(\alpha_s^2) \right)
\]

- Born
- Sudakov
- Approximate Virtual + Unresolved Real

**NLO Correction:** Subtract and correct by difference

\[
\begin{align*}
\frac{2 \text{Re}[M_0^0 M_0^{1*}]}{|M_0^0|^2} &= \frac{\alpha_s}{2\pi} 2C_F \left( 2I_{q\bar{q}}(\mu^2/m_Z^2) - 4 \right) \\
\int_0^s d\Phi_{\text{ant}} 2C_F g_s^2 A_{g/q\bar{q}} &= \frac{\alpha_s}{2\pi} 2C_F \left( -2I_{q\bar{q}}(\mu^2/m_Z^2) + \frac{19}{4} \right) \\
\end{align*}
\]

IR Singularity Operator
Getting Serious: second order

**Fixed Order:** Exclusive 3-jet rate (3 and only 3 jets), at $Q = Q_{\text{had}}$

$$\text{Exact} \quad \rightarrow \quad |M_1^0|^2 + 2 \text{Re}[M_1^0 M_1^{1*}] + \int_0^{Q_{\text{had}}^2} \frac{d\Phi_2}{d\Phi_1} |M_2^0|^2$$

**Vincia:**

2→3 Evolution Step

3→4 Evolution Step

Approximate $\rightarrow (1 + V_0) |M_1^0|^2 \Delta_2(m_Z^2, Q_1^2) \Delta_3(Q_{R1}^2, Q_{\text{had}}^2)$,

$V_0 = \frac{\alpha_s}{\pi}$

$\mu_R$
Loop Corrections

NLO Correction: Subtract and correct by difference

\[ V_{1Z}(q, g, \bar{q}) = \frac{2 \text{Re}[M_1^0 M_1^{1*}]}{|M_1^0|^2} \]^{\text{LC}} - \frac{\alpha_s}{\pi} - \frac{\alpha_s}{2\pi} \left( \frac{11N_C - 2n_F}{6} \right) \ln \left( \frac{\mu_{\text{ME}}^2}{\mu_{\text{PS}}^2} \right)

+ \frac{\alpha_s C_A}{2\pi} \left[ -2I_{qq}^{(1)}(\epsilon, \mu^2/s_{qg}) - 2I_{qg}^{(1)}(\epsilon, \mu^2/s_{qg}) + \frac{34}{3} \right]

+ \frac{\alpha_s n_F}{2\pi} \left[ -2I_{qg, F}^{(1)}(\epsilon, \mu^2/s_{qg}) - 2I_{gq, F}^{(1)}(\epsilon, \mu^2/s_{qg}) - 1 \right]

+ \frac{\alpha_s C_A}{2\pi} \left[ 8\pi^2 \int_{Q_1^2}^{m_Z^2} d\Phi_{\text{ant}} A_{g/q\bar{q}}^{\text{std}} + 8\pi^2 \int_{Q_1^2}^{m_Z^2} d\Phi_{\text{ant}} \delta A_{g/q\bar{q}} \right]

- \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} (1 - O_{Ej}) A_{g/qg}^{\text{std}} + \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} \delta A_{g/qg}

+ \frac{\alpha_s n_F}{2\pi} \left[ -\sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} (1 - O_{Sj}) P_{A_j} A_{\bar{q}/qg}^{\text{std}} + \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} \delta A_{\bar{q}/qg} \right]

- \frac{1}{6} \frac{s_{qg} - s_{g\bar{q}}}{s_{qg} + s_{g\bar{q}}} \ln \left( \frac{s_{qg}}{s_{g\bar{q}}} \right) \]
NLO Correction: Subtract and correct by difference

\[ V_{1Z}(q, g, \bar{q}) = \left[ \frac{2 \Re[M_1^0 M_1^*]}{|M_1^0|^2} \right]^{\text{LC}} - \frac{\alpha_s}{\pi} V_0 - \frac{\alpha_s}{2\pi} \left( \frac{11N_C - 2n_F}{6} \right) \ln \left( \frac{\mu_{\text{ME}}^2}{\mu_{\text{PS}}^2} \right) \]

\[ + \frac{\alpha_s C_A}{2\pi} \left[ -2I_{qq}^{(1)}(\epsilon, \mu^2/s_{qq}) - 2I_{qg}^{(1)}(\epsilon, \mu^2/s_{qg}) + \frac{34}{3} \right] \]

\[ + \frac{\alpha_s n_F}{2\pi} \left[ -2I_{qg,F}^{(1)}(\epsilon, \mu^2/s_{qq}) - 2I_{gq,F}^{(1)}(\epsilon, \mu^2/s_{qg}) - 1 \right] \]

\[ + \frac{\alpha_s C_A}{2\pi} \left[ 8\pi^2 \int_{Q_1^2}^{m_Z^2} d\Phi_{\text{ant}} A_{g/q\bar{q}}^{\text{std}} + 8\pi^2 \int_{Q_1^2}^{m_Z^2} d\Phi_{\text{ant}} \delta A_{g/q\bar{q}} \right] \]

\[ - \sum_{j=1}^{2} 8\pi^2 \int_0^{s_j} d\Phi_{\text{ant}} (1 - O_{Ej}) A_{g/qg}^{\text{std}} + \sum_{j=1}^{2} 8\pi^2 \int_0^{s_j} d\Phi_{\text{ant}} \delta A_{g/qg} \]

\[ + \frac{\alpha_s n_F}{2\pi} \left[ - \sum_{j=1}^{2} 8\pi^2 \int_0^{s_j} d\Phi_{\text{ant}} (1 - O_{Sj}) P_{Aj} A_{\bar{q}g/qg}^{\text{std}} + \sum_{j=1}^{2} 8\pi^2 \int_0^{s_j} d\Phi_{\text{ant}} \delta A_{\bar{q}g/qg} \right] \]

\[ - \frac{1}{6} \frac{s_{qq} - s_{g\bar{q}}}{s_{qq} + s_{g\bar{q}}} \ln \left( \frac{s_{qq}}{s_{g\bar{q}}} \right) \]

\[ \text{Q}_1 = 3\text{-parton Resolution Scale} \]

\[ \text{O}_{Ej} = \text{Gluon-Emission Ordering Function} \]

\[ \text{O}_{Sj} = \text{Gluon-Splitting Ordering Function} \]

The “Ariadne” Log

(72)

Hartgring, Laenen, Skands, arXiv:1303.4974
NLO Correction: Subtract and correct by difference

\[
V_{1Z}(q, g, \bar{q}) = \left[ \frac{2 \text{Re}[M_1^0 M_1^{*}]}{|M_1^0|^2} \right]^{\text{LC}} - \frac{\alpha_s}{\pi} V_0 - \frac{\alpha_s}{2\pi} \left( \frac{11N_C - 2n_F}{6} \right) \ln \left( \frac{\mu_{\text{ME}}^2}{\mu_{\text{PS}}^2} \right) \\
+ \frac{\alpha_s C_A}{2\pi} \left[ -2I_{qg}^{(1)}(\epsilon, \mu^2/s_{qg}) - 2I_{qg}^{(1)}(\epsilon, \mu^2/s_{\bar{q}g}) + \frac{34}{3} \right] \\
+ \frac{\alpha_s n_F}{2\pi} \left[ -2I_{qg,F}^{(1)}(\epsilon, \mu^2/s_{qg}) - 2I_{qg,F}^{(1)}(\epsilon, \mu^2/s_{qg}) - 1 \right] \\
+ \frac{\alpha_s C_A}{2\pi} \left[ 8\pi^2 \int_{Q_1^2}^{m_Z^2} \text{d}\Phi_{\text{ant}} \ A_{g/q\bar{q}}^\text{std} + 8\pi^2 \int_{Q_1^2}^{m_Z^2} \text{d}\Phi_{\text{ant}} \ \delta A_{g/q\bar{q}} \right. \\
- \left. \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} \text{d}\Phi_{\text{ant}} \ (1 - O_{E_j}) A_{g/qg}^\text{std} + \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} \text{d}\Phi_{\text{ant}} \ \delta A_{g/qg} \right] \\
+ \frac{\alpha_s n_F}{2\pi} \left[ -\sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} \text{d}\Phi_{\text{ant}} \ (1 - O_{S_j}) P_{A_j} A_{qg}^\text{std} + \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} \text{d}\Phi_{\text{ant}} \ \delta A_{\bar{q}/qg} \right. \\
- \left. \frac{1}{6} \frac{s_{qg} - s_{g\bar{q}}}{s_{qg} + s_{g\bar{q}}} \ln \left( \frac{s_{qg}}{s_{g\bar{q}}} \right) \right].
\]
**NLO Correction:** Subtract and correct by difference

\[ V_{1Z}(q, g, \bar{q}) = \left[ \frac{2 \text{Re}[M_1^0 M_1^*]}{|M_1^1|^2} \right]^{\text{LC}} - \frac{\alpha_s}{2\pi} V_0 - \frac{\alpha_s}{2\pi} \left( \frac{11N_C - 2n_F}{6} \right) \ln\left( \frac{\mu_R^2}{\mu_{\text{ME}}^2} \right) \]

\[ + \frac{\alpha_s C_A}{2\pi} \left[ -2I_{gq,1}^{(1)}(\epsilon, \mu^2/s_{gq}) - 2I_{qg,1}^{(1)}(\epsilon, \mu^2/s_{gq}) + \frac{34}{3} \right] \]

\[ + \frac{\alpha_s n_F}{2\pi} \left[ -2I_{gq,F,1}^{(1)}(\epsilon, \mu^2/s_{gq}) - 2I_{qg,F,1}^{(1)}(\epsilon, \mu^2/s_{gq}) - 1 \right] \]

\[ + \frac{\alpha_s C_A}{2\pi} \left[ 8\pi^2 \int_{Q_1^2}^{m_2^2} \text{d}\Phi_{\text{ant}} A_{g/\bar{q}}^{\text{std}} + 8\pi^2 \int_{Q_1^2}^{m_2^2} \text{d}\Phi_{\text{ant}} \delta A_{g/\bar{q}} \right] 
\]

\[ - \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} \text{d}\Phi_{\text{ant}} (1 - O_{E,j}) A_{g/\bar{q}}^{\text{std}} + \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} \text{d}\Phi_{\text{ant}} \delta A_{g/\bar{q}} \]

\[ + \frac{\alpha_s n_F}{2\pi} \left[ - \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} \text{d}\Phi_{\text{ant}} (1 - O_{S,j}) P_{A_j} A_{\bar{q}/g}^{\text{std}} + \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} \text{d}\Phi_{\text{ant}} \delta A_{\bar{q}/g} \right] \]

\[ - \frac{1}{6} \frac{s_{qg} - s_{g\bar{q}}}{s_{qg} + s_{g\bar{q}}} \ln \left( \frac{s_{qg}}{s_{g\bar{q}}} \right) \]

\[ (72) \]
NLO Correction: Subtract and correct by difference

\[ V_{1Z}(q, g, q) = \left[ \frac{2 \text{Re}[M_1^0 M_1^{0*}]}{|M_1^0|^2} \right]^{\text{LC}} - \frac{\alpha_s}{\pi} - \frac{\alpha_s}{2\pi} \left( \frac{11N_C - 2n_F}{6} \right) \ln \left( \frac{\mu^2_{\text{ME}}}{\mu^2_{\text{PS}}} \right) \]

Gluon Emission IR Singularity (std antenna integral)

\[ + \frac{\alpha_s C_A}{2\pi} \left[ -2I_{qg}^{(1)}(\epsilon, \mu^2/s_{qg}) - 2I_{qg}^{(1)}(\epsilon, \mu^2/s_{gq}) + \frac{34}{3} \right] \]

Gluon Splitting IR Singularity (std antenna integral)

\[ + \frac{\alpha_s n_F}{2\pi} \left[ -2I_{qg,F}^{(1)}(\epsilon, \mu^2/s_{qg}) - 2I_{gq,F}^{(1)}(\epsilon, \mu^2/s_{gq}) - 1 \right] \]

Standard (universal) 2→3 Sudakov Logs

\[ + \frac{\alpha_s C_A}{2\pi} \left[ 8\pi^2 \int_{Q_1^2}^{m_Z^2} d\Phi_{\text{ant}} A_{g/qg}^{\text{std}} + 8\pi^2 \int_{Q_1^2}^{m_Z^2} d\Phi_{\text{ant}} \delta A_{g/qg} \right] \]

\[ - \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} (1 - O_{Ej}) A_{g/qg}^{\text{std}} + \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} \delta A_{g/qg} \]

\[ + \frac{\alpha_s n_F}{2\pi} \left[ - \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} (1 - O_{Sj}) P_{Aj} A_{qg/qg}^{\text{std}} + \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} \delta A_{qg/qg} \right] \]

\[ - \frac{1}{6} \frac{s_{qg} - s_{gq}}{s_{qg} + s_{gq}} \ln \left( \frac{s_{qg}}{s_{gq}} \right) \]

\[ \text{The “Ariadne Log”} \]
**NLO Correction:** Subtract and correct by difference

\[
V_{1Z}(q, g, \bar{q}) = \left[ \frac{2 \text{Re}[M_1^0 M_1^*]}{|M_1^1|^2} \right]^{\text{LC}} - \frac{\alpha_s}{\pi} V_0 - \frac{\alpha_s}{2\pi} \left( \frac{11N_C - 2n_F}{6} \right) \ln \left( \frac{\mu_{\text{ME}}^2}{\mu_{\text{PS}}^2} \right) + \frac{\alpha_s C_A}{2\pi} \left[ -2I_{qq}^{(1)}(\epsilon, \mu^2/s_{qg}) - 2I_{qq}^{(1)}(\epsilon, \mu^2/s_{q\bar{q}}) + \frac{34}{3} \right] \\
+ \frac{\alpha_s n_F}{2\pi} \left[ -2I_{qg,F}^{(1)}(\epsilon, \mu^2/s_{qg}) - 2I_{qg,F}^{(1)}(\epsilon, \mu^2/s_{q\bar{q}}) - 1 \right] \\
+ \frac{\alpha_s C_A}{2\pi} \left[ 8\pi^2 \int_{Q_1^2}^{m_Z^2} d\Phi_{\text{ant}} A_{g/q\bar{q}}^{\text{std}} + 8\pi^2 \int_{Q_1^2}^{m_Z^2} d\Phi_{\text{ant}} \delta A_{g/q\bar{q}} \right] \\
- \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} (1 - O_{Ej}) A_{g/qg}^{\text{std}} + \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} \delta A_{g/qg} \\
+ \frac{\alpha_s n_F}{2\pi} \left[ -2 \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} (1 - O_{Sj}) P_{Aj} A_{q\bar{q}g}^{\text{std}} + \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} \delta A_{q\bar{q}g} \right] \\
- \frac{1}{6} \frac{s_{qg} - s_{q\bar{q}}}{s_{qg} + s_{q\bar{q}}} \ln \left( \frac{s_{qg}}{s_{q\bar{q}}} \right),
\]

\( Q_1 = 3 \)-parton Resolution Scale

\( O_{Ej} = \text{Gluon-Emission Ordering Function} \)

\( O_{Sj} = \text{Gluon-Splitting Ordering Function} \)

The “Ariadne” Log

(72)
**NLO Correction:** Subtract and correct by difference

\[
V_{1Z}(q, g, \bar{q}) = \left[ \frac{2 \text{Re}[M_1^0 M_1^{1*}]}{|M_1|^2} \right]^{\text{LC}} - \frac{\alpha_s V_0}{2\pi} - \frac{\alpha_s}{2\pi} \left( \frac{11N_C - 2n_F}{6} \right) + \frac{\mu_R}{\mu_{\text{ME}}} \ln \left( \frac{\mu_{\text{ME}}^2}{\mu_{\text{PS}}^2} \right)
\]

\[
+ \frac{\alpha_s C_A}{2\pi} \left[ -2I_{qq}^{(1)}(\epsilon, \mu^2/s_{qq}) - 2I_{qg}^{(1)}(\epsilon, \mu^2/s_{qg}) + \frac{34}{3} \right]
\]

\[
+ \frac{\alpha_s n_F}{2\pi} \left[ -2I_{gq,F}^{(1)}(\epsilon, \mu^2/s_{gq}) - 2I_{gq,F}^{(1)}(\epsilon, \mu^2/s_{gq}) - 1 \right]
\]

\[
+ \frac{\alpha_s C_A}{2\pi} \left[ 8\pi^2 \int_{Q_1^2}^{m_Z^2} d\Phi_{\text{ant}} A_{g/q\bar{q}}^{\text{std}} + 8\pi^2 \int_{Q_1^2}^{m_Z^2} d\Phi_{\text{ant}} \delta A_{g/q\bar{q}} \right]
\]

\[
- \sum_{j=1}^{2} \left[ 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} (1 - O_{Ej}) A_{g/q\bar{q}}^{\text{std}} + 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} \delta A_{g/q\bar{q}} \right]
\]

\[
+ \frac{\alpha_s n_F}{2\pi} \left[ - \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} (1 - O_{Sj} P_{Aj}) A_{\bar{q}/g\bar{q}}^{\text{std}} + 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} \delta A_{\bar{q}/g\bar{q}} \right]
\]

\[
- \frac{1}{6} \frac{s_{qg} - s_{gq}}{s_{qg} + s_{gq}} \ln \left( \frac{s_{qg}}{s_{gq}} \right)
\]

**Gluon Emission IR Singularity** (std antenna integral)

**Gluon Splitting IR Singularity** (std antenna integral)

**Standard (universal) 2→3 Sudakov Logs**

**Standard (universal) 3→4 Sudakov Logs: C_A**

**Standard (universal) 3→4 Sudakov Logs: n_F**

appendix of our paper + functions in the code
\[ V_{1Z}(q, g, \bar{q}) = \left[ \frac{2 \operatorname{Re}[M_1^0 M_1^{1*}]}{|M_1^1|^2} \right]_{\text{LC}}^{\text{ME}} - \frac{\alpha_s V_0}{\pi} \left[ \frac{11 N_C - 2 n_F}{6} \right] \ln \left( \frac{\mu_{\text{MF}}^2}{\mu_{\text{PS}}^2} \right) \]

\[ + \frac{\alpha_s C_A}{2\pi} \left[ -2I_{qg}^{(1)}(\epsilon, \mu^2/s_{qg}) - 2I_{qg}^{(1)}(\epsilon, \mu^2/s_{\bar{q}\bar{q}}) + \frac{34}{3} \right] \]

\[ + \frac{\alpha_s n_F}{2\pi} \left[ -2I_{qg,F}^{(1)}(\epsilon, \mu^2/s_{qg}) - 2I_{qg,F}^{(1)}(\epsilon, \mu^2/s_{qg}) - 1 \right] \]

- \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} (1 - O_{Sj}) A_{g/qg}^{\text{std}} + \sum_{j=1}^{2} 8\pi^2 \int_{0}^{s_j} d\Phi_{\text{ant}} \delta A_{g/qg}^{\text{ME}/\text{PS}} 

\[ \delta A_{g/qg}^{\text{ME}/\text{PS}} = \text{Integrals over ME/PS corrections} \]

\[ \text{Done numerically} \]

\[ Q_1 = 3\text{-parton Resolution Scale} \]

\[ O_{Ej} = \text{Gluon-Emission Ordering Function} \]

\[ O_{Sj} = \text{Gluon-Splitting Ordering Function} \]

\[ \frac{1}{6} \frac{s_{qg} - s_{\bar{q}\bar{q}}}{s_{qg} + s_{\bar{q}\bar{q}}} \ln \left( \frac{s_{qg}}{s_{\bar{q}\bar{q}}} \right) \]

The “Ariadne Log”
energy-ordering variables intersect the phase-space boundaries, where the antenna functions are singu-
liners of invariants, for mass-ordering (\text{Figure 2}:
\begin{align*}
Q_E^2 &= 4p_{\perp}^2 = \frac{4s_{ij}s_{jk}}{s_{ijk}}.
\end{align*}
Note that, for the special case of the
\text{2→3:}
\begin{align*}
a_3^0 &= \frac{1}{s} \left( \frac{2y_{ik}}{y_{ij}y_{jk}} + \frac{y_{ij}}{y_{jk}} + \frac{y_{jk}}{y_{ij}} \right) \\
&= \frac{\alpha_s C_A}{2\pi} \left( \sum_{i=1}^{5} K_i I_i(s, Q_3^2) \right) \\
K_1 &= 1, \quad K_2 = -2, \quad K_3 = 2, \quad K_4 = -\delta_{Ig} - \delta_{Kg}, \quad K_5 = 1.
\end{align*}
\begin{align*}
I_1 &= -\text{Li}_2 \left( \frac{1}{2} \left( 1 + \sqrt{1 - y_3^2} \right) \right) + \text{Li}_2 \left( \frac{1}{2} \left( 1 - \sqrt{1 - y_3^2} \right) \right) - \frac{1}{2} \ln \left( \frac{4}{y_3^2} \right) \ln \left( \frac{1 - \sqrt{1 - y_3^2}}{1 + \sqrt{1 - y_3^2}} \right) \\
I_2 &= -2\sqrt{1 - y_3^2} + \ln \left( \frac{1 + \sqrt{1 - y_3^2}}{1 - \sqrt{1 - y_3^2}} \right) \\
I_3 &= -\frac{1}{4} \sqrt{1 - y_3^2} + \frac{1}{4} \ln \left( \frac{1 + \sqrt{1 - y_3^2}}{1 - \sqrt{1 - y_3^2}} \right) \\
I_4 &= \left[ -\frac{13\sqrt{1 - y_3^2}}{36} + \frac{1}{36} y_3^2 \sqrt{1 - y_3^2} + \frac{3}{3} \ln \left( 1 + \sqrt{1 - y_3^2} \right) - \ln \left( y_3^2 \right) - \frac{3}{6} \right] \\
I_5 &= \frac{1}{24} \left[ 2 \left( 3C_{00} - (C_{01} + C_{10})(-1 + y_3^2) \sqrt{1 - y_3^2} - 3C_{00} y_3 \ln \left( \frac{1 + \sqrt{1 - y_3^2}}{1 - \sqrt{1 - y_3^2}} \right) \right) \right].
\end{align*}
\text{3→4: } C_A \text{ piece (for strong ordering)}
\begin{align*}
-g_s^2 \sum_{j=1}^{2} C_A \int_{0}^{s_j} (1 - O_{E_j}) \, d\Phi_{\text{ant}} &= -\frac{\alpha_s C_A}{2\pi} \left( \sum_{i=1}^{5} K_i I_i(s_{qg}, Q_3^2) \right) - \frac{\alpha_s C_A}{2\pi} \left( \sum_{i=1}^{5} K_i I_i(s_{qg}, Q_3^2) \right)
\end{align*}
The $\delta A$ Terms - Speed

Figure 14: Distribution of the size of the $\delta A$ terms (normalized so the LO result is unity) in actual VINCIA runs. Left: linear scale, default settings. Right: logarithmic scale, with variations on the minimum number of MC points used for the integrations (default is 100).

<table>
<thead>
<tr>
<th>LO level</th>
<th>NLO level</th>
<th>Time / Event [milliseconds]</th>
<th>Speed relative to PYTHIA $1_{\text{Time}} / \text{PYTHIA 8}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z \rightarrow Z$</td>
<td>2, 3</td>
<td>0.4</td>
<td>1</td>
</tr>
<tr>
<td>VINCIA (NLO off)</td>
<td>2, 3, 4, 5</td>
<td>2.2</td>
<td>$\sim 1/5$</td>
</tr>
<tr>
<td>VINCIA (NLO on)</td>
<td>2, 3, 4, 5</td>
<td>3.0</td>
<td>$\sim 1/7$</td>
</tr>
</tbody>
</table>

**Speed:**

- **Default Settings**
- **nMC = 100**
- **nMC = 400**
- **nMC = 1600**

Hartgring, Laenen, Skands, arXiv:1303.4974
1) IR Limits

Pole-subtracted one-loop matrix element

\[
S_{\text{Virtual}} = \left[ \frac{2 \text{Re}[M_3^{0} M_3^{1*}]}{|M_3^{0}|^2} \right]_{\text{LC}} + \frac{\alpha_s C_A}{2\pi} \left[ -2I_{qq}^{(1)}(\epsilon, \mu^2/s_{qq}) - 2I_{qg}^{(1)}(\epsilon, \mu^2/s_{qg}) + \frac{34}{3} \right] + \frac{\alpha_s n_F}{2\pi} \left[ -2I_{qg,F}^{(1)}(\epsilon, \mu^2/s_{qg}) - 2I_{gq,F}^{(1)}(\epsilon, \mu^2/s_{qg}) - 1 \right]
\]

\[
S_{\text{Virtual}} \begin{array}{l|l}
\text{soft} & \left( -L^2 - \frac{10}{3} L - \frac{\pi^2}{6} \right) C_A + \frac{1}{3} n_F L \\
\text{hard collinear} & -\frac{5}{3} L C_A + \frac{1}{6} n_F L
\end{array}
\]

\[s_{qq} = s_{g\bar{q}} = y \to 0\]

\[s_{qq} = y \to 0, s_{g\bar{q}} \to s\]

Second-Order Antenna Shower Expansion:

<table>
<thead>
<tr>
<th>$p_\perp$</th>
<th>strong</th>
<th>smooth</th>
<th>$V_{3Z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>soft</td>
<td>( \left( L^2 - \frac{1}{3} L + \frac{\pi^2}{6} \right) C_A + \frac{1}{3} n_F L )</td>
<td>( \left( L^2 - \frac{1}{3} L - \frac{\pi^2}{6} \right) C_A + \frac{1}{3} n_F L )</td>
<td>$-\beta_0 L$</td>
</tr>
<tr>
<td>hard collinear</td>
<td>$-\frac{1}{6} L C_A + \frac{1}{6} n_F L$</td>
<td>( -\frac{1}{6} L - \frac{\pi^2}{6} ) C_A + \frac{1}{6} n_F L</td>
<td>$-\frac{1}{2} \beta_0 L$</td>
</tr>
<tr>
<td>$m_D$</td>
<td>soft</td>
<td>( \left( L^2 + \frac{3}{2} L - \frac{\pi^2}{6} \right) C_A )</td>
<td>( \left( L^2 + \frac{3}{2} L - \frac{\pi^2}{6} \right) C_A )</td>
</tr>
<tr>
<td>hard collinear</td>
<td>$-\frac{1}{6} L C_A + \frac{1}{6} n_F L$</td>
<td>( -\frac{1}{6} L - \frac{\pi^2}{3} ) C_A + \frac{1}{6} n_F L</td>
<td>$-\frac{1}{2} \beta_0 L$</td>
</tr>
</tbody>
</table>
2) NLO Evolution

**Vincia**: NLO $Z \to 2 \to 3$ Jets + Markov Shower

**Size of NLO Correction:**
over 3-parton Phase Space

**Markov Evolution in:**
Transverse Momentum

**Parameters:**

\[
\alpha_s(M_Z) = 0.12
\]
\[
\mu_R = m_Z
\]
\[
\Lambda_{QCD} = \Lambda_{MS}
\]

**Scaled Invariants**

\[
y_{ij} = \frac{2(p_i \cdot p_j)}{M_Z^2}
\]
→ 0 when $i \parallel j$
& when $E_j \to 0$

Hartgring, Laenen, Skands, arXiv:1303.4974
Renormalization: 1) Choose $\mu_R \sim p_{Tjet}$ (absorbs universal $\beta$-dependent terms)
2) Translate from MSbar to CMW scheme ($\Lambda_{CMW} \sim 1.6 \Lambda_{MSbar}$ for coherent showers)

Markov Evolution in: Transverse Momentum, $\alpha_S(M_Z) = 0.12$
The choice of evolution variable ($Q$)

**Parameters:** $\alpha_s(M_Z) = 0.12$, $\mu_R = p_{Tg}$

**Figure 6:** NLO correction factor for strong

**Figure 2:**

**Evolution in Dipole Mass**

**Evolution in $p_T$**

- Missing Sudakov Suppression in Soft Region
- Too much Sudakov Suppression in Collinear Region
- Small Corrections Everywhere
The proof of the pudding

LO Tunes
(both VINCIA and PYTHIA)

$$\alpha_s(M_Z)^{\text{MSbar}} \sim 0.139$$

(LO matrix elements give similar values, and also LO PDFs)

**New VINCIA NLO Tune**

$$\alpha_s(M_Z)^{\text{CMW}} = 0.122$$

with 2-loop running (new)

<table>
<thead>
<tr>
<th>$\langle \chi^2 \rangle$ Shapes</th>
<th>$T$</th>
<th>$C$</th>
<th>$D$</th>
<th>$B_W$</th>
<th>$B_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PYTHIA 8</td>
<td>0.4</td>
<td>0.4</td>
<td>0.6</td>
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<td>0.2</td>
</tr>
<tr>
<td>VINCIA (LO)</td>
<td>0.2</td>
<td>0.4</td>
<td>0.4</td>
<td>0.3</td>
<td>0.3</td>
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<td>0.2</td>
<td>0.6</td>
<td>0.3</td>
<td>0.2</td>
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<table>
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<tr>
<th>$\langle \chi^2 \rangle$ Frag</th>
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<th>$x$</th>
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<th>Baryons</th>
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<tr>
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<td>0.7</td>
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<td>0.6</td>
</tr>
</tbody>
</table>

Data from Phys.Rept. 399 (2004) 71

Hartgring, Laenen, Skands, arXiv:1303.4974
Beyond Perturbation Theory

Better pQCD → Better non-perturbative constraints

Soft QCD & Hadronization:
Less perturbative ambiguity → improved clarity

ALICE/RHIC:
pp as reference for AA
Collective (soft) effects in pp

Pb+Pb @ sqrt(s) = 2.76 ATeV

2010-11-08 11:29:42
Fill : 1482
Run : 137124
Event : 0x00000000271EC693

central slice
(0.5% of tracks in the event)
Beyond Colliders?

Other uses for a high-precision fragmentation model

Dark-matter annihilation:
Photon & particle spectra

Cosmic Rays:
Extrapolations to ultra-high energies

ISS, March 28, 2012
Aurora and sunrise over Ireland & the UK
Thank You

Outlook

p → p

Thank You
+ 2nd order showers
NLO ee → 4 jets
NLO w helicity dependence
NLO w massive fermions
NLO automated
Interleaved showers & dec.
Niels Erik
3 Months Today
Fixed Order: Recap

**Improve by** computing quantum corrections, order by order


**Leading Order**

<table>
<thead>
<tr>
<th>k (legs)</th>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
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<tbody>
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<td>$\sigma_1^{(0)}$</td>
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<td>2</td>
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<td>...</td>
</tr>
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</table>

Max Born, 1882-1970
Nobel 1954

**Next-to-Leading Order**

<table>
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<tr>
<th>k (legs)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
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</thead>
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</tbody>
</table>
Fixed Order: Recap

**Improve by** computing quantum corrections, order by order

(From PS, Introduction to QCD, TASI 2012, arXiv:1207.2389)

**Leading Order**

\[
\sigma^{\text{NLO}} = \sigma^{\text{Born}} + \int d\Phi_{F+1} \left| M_{F+1}^{(0)} \right|^2 + \int d\Phi_F 2\text{Re} \left[ M_F^{(1)} M_F^{(0)\ast} \right]
\]

\[
\rightarrow 1/\epsilon^2 + 1/\epsilon + \text{Finite}
\]

**Next-to-Leading Order**

Max Born, 1882-1970
Nobel 1954

\[
\sigma^{\text{NLO}} = \sigma^{\text{Born}} + \int d\Phi_{F+1} \left| M_{F+1}^{(0)} \right|^2 + \int d\Phi_F 2\text{Re} \left[ M_F^{(1)} M_F^{(0)\ast} \right]
\]

\[
\rightarrow -1/\epsilon^2 - 1/\epsilon + \text{Finite}
\]
**Fixed Order: Recap**

*Improve by* computing quantum corrections, order by order


---

**Leading Order**

\[
\sigma^{\text{NLO}} = \sigma^{\text{Born}} + \int d\Phi_{F+1} \left| M^{(0)}_{F+1} \right|^2 + \int d\Phi_F \text{Re} \left[ M^{(1)}_F M^{(0)*}_F \right]
\]

**Next-to-Leading Order**

\[
\sigma^{\text{NLO}} = \sigma^{\text{Born}} + \int d\Phi_{F+1} \left| M^{(0)}_{F+1} \right|^2 + \int d\Phi_F \text{Re} \left[ M^{(1)}_F M^{(0)*}_F \right] + \int d\Phi_{F+1} d\sigma^{\text{NLO}}_S
\]

---

**The Subtraction Idea**

Universal "Subtraction Terms" (will return to later)

Finite by Universality

Finite by KLN
Figure 2: Schematic overview of how the full collinear singularity of parton $I$ and the soft singularity of the $IK$ pair, respectively, originate in different shower types. ($\Theta_I$ and $\Theta_K$ represent angular vetos with respect to partons $I$ and $K$, respectively, and $\Theta_{IK}$ represents a sector phase-space veto, see text.)
### Global Antennae

Table 2: Table of coefficients for helicity-dependent global antenna functions. By the $C$ and $P$ invariance of QCD, the same expressions apply with $+\bar{q}$, $q\bar{q}$. All other antennae are zero. The parameter $\alpha$ determines the form of the spin-summed global antennae. The default choice in VINCIA is $\alpha = 0$ which corresponds to the GGG spin-summed antennae. The finite terms are chosen so that the antennae are positive on all of final state phase space.

<table>
<thead>
<tr>
<th>$\times$</th>
<th>$\frac{1}{y_{ij}y_{jk}}$</th>
<th>$\frac{1}{y_{ij}}$</th>
<th>$\frac{1}{y_{jk}}$</th>
<th>$\frac{y_{jk}}{y_{ij}}$</th>
<th>$\frac{y_{ij}}{y_{jk}}$</th>
<th>$\frac{y_{jk}^2}{y_{ij}}$</th>
<th>$\frac{y_{ij}^2}{y_{jk}}$</th>
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</table>
### Sector Antennae

**Global**

\[
\bar{a}^{gl}_{g/qg}(p_i, p_j, p_k) \xrightarrow{s_{jk} \to 0} \frac{1}{s_{jk}} \left( P_{gg \to G}(z) - \frac{2z}{1 - z} - z(1 - z) \right)
\]

\( \rightarrow P(z) = \text{Sum over two neighboring antennae} \)

**Sector**

Only a single term in each phase space point

\[
\bar{a}_{j/IK}^{\text{sct}}(y_{ij}, y_{jk}) = \bar{a}_{j/IK}^{\text{gl}}(y_{ij}, y_{jk}) \quad + \quad \delta_{Ig} \delta_{HK} H_k \left\{ \delta_{H_j H_i} \delta_{H_1 H_j} \left( \frac{1 + y_{jk} + y_{jk}^2}{y_{ij}} \right) \right. \\
\left. + \quad \delta_{H_1 H_j} \left( \frac{1}{y_{ij}(1 - y_{jk})} - \frac{1 + y_{jk} + y_{jk}^2}{y_{ij}} \right) \right\} \\
+ \quad \delta_{Kg} \delta_{H_1 H_i} \left\{ \delta_{H_1 H_j} \delta_{HK H_k} \left( \frac{1 + y_{ij} + y_{ij}^2}{y_{jk}} \right) \right. \\
\left. + \quad \delta_{HK H_j} \left( \frac{1}{y_{jk}(1 - y_{ij})} - \frac{1 + y_{ij} + y_{ij}^2}{y_{jk}} \right) \right\}
\]

Sector = Global + additional collinear terms (from “neighboring” antenna)
In a traditional parton shower, you would face the following problem:

Existing parton showers are not really Markov Chains

Further evolution (restart scale) depends on which branching happened last → proliferation of terms

Number of histories contributing to \( n^{\text{th}} \) branching \( \propto 2^n n! \)

\[
\begin{array}{c}
\sim \\
+ \\
+ \\
+ \\
\end{array}
\]

\( j = 2 \)
\( \rightarrow 4 \text{ terms} \)

\[
\left( \begin{array}{c}
\sim \\
+ \\
\end{array} \right) \quad j = 1
\]
\( \rightarrow 2 \text{ terms} \)

Parton- (or Catani-Seymour) Shower:
After 2 branchings: 8 terms
After 3 branchings: 48 terms
After 4 branchings: 384 terms

(+ parton showers have complicated and/or frame-dependent phase-space mappings, especially at the multi-parton level)
**Antenna showers:** one term per parton pair

\[ 2^n n! \rightarrow n! \]

**+ Change “shower restart” to Markov criterion:**

Given an \( n \)-parton configuration, “ordering” scale is

\[ Q_{ord} = \min(Q_{E1}, Q_{E2}, \ldots, Q_{En}) \]

Unique restart scale, independently of how it was produced

**+ Matching: \( n! \rightarrow n \)**

Given an \( n \)-parton configuration, its phase space weight is:

\[ |M_n|^2 : \text{Unique weight, independently of how it was produced} \]

---

**Matched Markovian Antenna Shower:**
- After 2 branchings: 2 terms
- After 3 branchings: 3 terms
- After 4 branchings: 4 terms

**Parton- (or Catani-Seymour) Shower:**
- After 2 branchings: 8 terms
- After 3 branchings: 48 terms
- After 4 branchings: 384 terms

---

(+ generic Lorentz-invariant and on-shell phase-space factorization)

**Giele, Kosower, Skands, PRD 84 (2011) 054003**

Lopez-Villarejo, Skands, JHEP 1111 (2011) 150
Q: How well do showers do?

Exp: Compare to data. Difficult to interpret; all-orders cocktail including hadronization, tuning, uncertainties, etc

Th: Compare products of splitting functions to full tree-level matrix elements

Plot distribution of $\log_{10}(PS/ME)$

- $Z \rightarrow 4$
  - Vincia 1.025 + MadGraph 4.426
  - Matched to $Z \rightarrow 3$
  - Strong Ordering

- $Z \rightarrow 5$
  - Vincia 1.025 + MadGraph 4.426
  - Matched to $Z \rightarrow 3$
  - Strong Ordering

- $Z \rightarrow 6$
  - Vincia 1.025 + MadGraph 4.426
  - Matched to $Z \rightarrow 3$
  - Strong Ordering

Dead Zone: 1-2% of phase space have no strongly ordered paths leading there.*

*fine from strict LL point of view: those points correspond to “unordered” non-log-enhanced configurations
Generate Branchings without imposing strong ordering

At each step, each dipole allowed to fill its entire phase space

Overcounting removed by matching
+ smooth ordering beyond matched multiplicities

\[
\frac{p_\perp^2}{\hat{p}_\perp^2 + p_\perp^2} \quad P_{LL} \quad \frac{p_\perp^2}{p_\perp^2} \quad \text{last branching}
\]
\[
\frac{p_\perp^2}{p_\perp^2} \quad \text{current branching}
\]
→ Better Approximations

Distribution of $\log_{10}(PS_{LO}/ME_{LO})$ (inverse ~ matching coefficient)

Z→ 4
- Vincia 1.025 + MadGraph 4.426
- Matched to Z→ 3
- Strong Ordering

Z→ 5
- Vincia 1.025 + MadGraph 4.426
- Matched to Z→ 3
- Strong Ordering

Z→ 6
- Vincia 1.025 + MadGraph 4.426
- Matched to Z→ 3
- Strong Ordering

Leading Order, Leading Color, Flat phase-space scan, over all of phase space (no matching scale)

Z→ 4
- Vincia 1.025 + MadGraph 4.426
- Matched to Z→ 3
- Smooth Ordering

Z→ 5
- Vincia 1.025 + MadGraph 4.426
- Matched to Z→ 3
- Smooth Ordering

Z→ 6
- Vincia 1.025 + MadGraph 4.426
- Matched to Z→ 3
- Smooth Ordering

No dead zone
+ Matching (+ full colour)

→ A very good all-orders starting point
IR Singularity Operators

\( q\bar{q} \to qg\bar{q} \) antenna function

\[
A^0_3(1_q, 3_g, 2_{\bar{q}}) = \frac{1}{s_{123}} \left( \frac{s_{13}}{s_{23}} + \frac{s_{23}}{s_{13}} + 2 \frac{s_{12}s_{13}}{s_{13}s_{23}} \right)
\]

Integrated antenna

\[
\mathcal{P}oles \left( A^0_3(s_{123}) \right) = -2I^{(1)}_{qq}(\epsilon, s_{123})
\]

\[
\mathcal{F}inite \left( A^0_3(s_{123}) \right) = \frac{19}{4}.
\]

\[
\chi^0_{ijk}(s_{ijk}) = (8\pi^2(4\pi)^{-\epsilon}e^{\epsilon\gamma}) \int d\Phi X_{ijk} X^0_{ijk}.
\]

Singularity Operators

\[
I^{(1)}_{qq}(\epsilon, \mu^2/s_{qq}) = -\frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)} \left[ \frac{1}{\epsilon^2} + \frac{3}{2\epsilon} \right] \text{Re} \left( -\frac{\mu^2}{s_{qq}} \right)^\epsilon
\]

\[
I^{(1)}_{qq}(\epsilon, \mu^2/s_{qq}) = -\frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)} \left[ \frac{1}{\epsilon^2} + \frac{5}{3\epsilon} \right] \text{Re} \left( -\frac{\mu^2}{s_{qq}} \right)^\epsilon
\]

\[
I^{(1)}_{qq,F}(\epsilon, \mu^2/s_{qq}) = \frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)} \frac{1}{6\epsilon} \text{Re} \left( -\frac{\mu^2}{s_{qq}} \right)^\epsilon
\]

for \( qg \to qg \)

for \( qg \to qq'q' \)
Uncertainties

No calculation is more precise than the reliability of its uncertainty estimate → aim for full assessment of TH uncertainties.
Another use for simple analytical expansions?

For each event, can compute probability this event would have resulted under alternative conditions.

+ **Unitarity**: also recompute no-evolution probabilities.

\[
P_2 = \frac{\alpha s_2 a_2}{\alpha s_1 a_1} P_1
\]

\[
P_{2;no} = 1 - P_2 = 1 - \frac{\alpha s_2 a_2}{\alpha s_1 a_1} P_1
\]
Traditional Approach:
Run calculation $1_{\text{central}} + 2N_{\text{variations}} = \text{slow}$

Another use for simple analytical expansions?

For each event, can compute probability this event would have resulted under alternative conditions

$$P_2 = \frac{\alpha_s^2 a_2}{\alpha_s^1 a_1} P_1$$

+ **Unitarity**: also recomputes no-evolution probabilities

$$P_{2;\text{no}} = 1 - P_2 = 1 - \frac{\alpha_s^2 a_2}{\alpha_s^1 a_1} P_1$$

VINCI A: = fast, automatic

Central weights = 1
+ N sets of alternative weights = variations (all with $<w>=1$)
→ For every configuration/event, calculation tells how sure it is
Bonus: events only have to be hadronized & detector-simulated ONCE!
Example of Physical Observable: **Before** (left) and **After** (right) Matching

Jet Broadening = LEP event-shape variable, measures “fatness” of jets
Example: Non-Singular Terms

Thrust = LEP event-shape variable, goes from 0 (pencil) to 0.5 (hedgehog)
Example: $\mu_R$

**Thrust** = LEP event-shape variable, goes from 0 (pencil) to 0.5 (hedgehog)